Non-Lorentzian Supergravity

非洛伦兹超引力

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埃里克·A·贝赫舒夫与扬·罗塞尔

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Abstract

摘要

We give an overview of the different non-Lorentzian supergravity theories in diverse dimensions that have been constructed in recent years. After giving a detailed discussion of non-Lorentzian geometries, as compared to Lorentzian geometries, we outline some of the construction methods that have been applied to obtain non-Lorentzian supergravity. Explicit results are given for non-Lorentzian supergravity theories in three and ten dimensions.

我们概述了近年来在不同维度下构造出的各类非洛伦兹超引力理论。在对比洛伦兹几何、详细讨论非洛伦兹几何后，我们梳理了目前用于构造非洛伦兹超引力的几种主要方法，并给出了三维和十维非洛伦兹超引力理论的明确结果。

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Introduction

引言

General relativity as the theory of classical relativistic gravity is a well-established framework already for more than a hundred years. It has confronted many experimental tests with flying colours. The combination of quantum mechanics and gravity is expected to lead to a new theory called quantum gravity that is valid up to Planck scales. Despite tremendous effort, no one has succeeded in constructing such a theory to date.

广义相对论作为经典相对论引力理论，是一个已经建立超过百年的成熟框架。它顺利通过了诸多实验检验。量子力学与引力的结合，有望产生一个在普朗克尺度下依然成立的新理论，即量子引力。尽管付出了巨大努力，迄今为止仍没有人成功构建出这样一个理论。

Newtonian gravity, being the precursor of general relativity, is only able to describe gravitational physics for velocities that are low compared to the speed of light and that involve weak gravitational fields. Moreover, Newtonian gravity is only valid in reference frames that are rectilinearly accelerating with respect to inertial frames. A non-relativistic (Since every physical theory can be called relativistic, be it with respect to Galilean, Bargmann or Lorentzian symmetries, we will in this review also often use the more apt word non-Lorentzian instead of non-relativistic, when referring to theories that feature symmetries that can be viewed as $c \to \infty$ limits of Lorentzian ones. Note however that the phrase non-Lorentzian is more generic and is also used, e.g., for so-called Carroll symmetries that arise as $c \to 0$ limits of Lorentzian ones.) (NR) version of general relativity, formulated in arbitrary reference frames, was constructed by Cartan 8 years after Einstein's discovery of general relativity [1]. This theory is called Newton-Cartan (NC) gravity and is based on a NR version of the Riemannian geometry underlying general relativity. This is a degenerate metric geometry with a co-dimension one foliation in space-like leaves and an absolute time direction connecting them. This geometry is called NC geometry, and the corresponding gravity theory is called NC gravity. It is a reformulation of Newtonian gravity in the sense that one can show that it is always possible to choose reference frames that are rectilinearly accelerated with respect to Newtonian inertial frames and in which one recovers Newtonian gravity.

牛顿引力是广义相对论的前身，仅能描述速度远小于光速、且引力场较弱情形下的引力物理。此外，牛顿引力仅在相对于惯性系做直线加速的参考系中成立。卡坦在爱因斯坦发现广义相对论的 8 年后，构建了一个可以在任意参考系中表述的非相对论 (由于所有物理理论都可以被称为相对论性的，区别只是遵从伽利略对称性、巴格曼对称性还是洛伦兹对称性，因此在本综述中，当我们讨论具有可视为洛伦兹对称性 $c \to \infty$ 极限的对称性的理论时，我们常会使用更恰当的“非洛伦兹”而非“非相对论”一词。但需注意，非洛伦兹一词更具通用性，例如它也可用于描述由洛伦兹对称性取 $c \to 0$ 极限得到的所谓卡罗尔对称性。(NR) 广义相对论。该理论被称为牛顿-卡坦 (NC) 引力，它基于广义相对论底层黎曼几何的 NR 版本，是一种退化度量几何，具有余维数为一的类空叶分叶，且由绝对时间方向连接这些叶分。该几何被称为 NC 几何，对应的引力理论被称为 NC 引力。它是对牛顿引力的重新表述，可证明该理论总能选出相对于牛顿惯性系做直线加速的参考系，在这些参考系中可以还原出牛顿引力。

NC gravity and geometry have attracted great attention in the recent literature due to a variety of reasons. A suitable generalization of it plays a crucial role in the original formulation of a NR version of string theory - a candidate theory of quantum gravity [2,3]. The possible formulation of such a NR string theory is tied up with the fundamental question whether combining gravity with quantum mechanics requires special relativity yes or no. The formulation of a consistent NR string theory moreover leads to the exciting question whether the holographic description of quantum gravity has an extension with NR gravity in the bulk and a NR conformal field theory at the boundary. Independent of this, starting with [4, 5], NC gravity also has found several applications to condensed matter physics. Using an effective field theory description in which NR symmetries are implemented geometrically, the gravitational fields are used in this context as response functions to obtain new insights into the non-perturbative properties of the model under study.

由于多种原因，NC 引力与几何近年来受到了研究界的广泛关注。对它的恰当推广在量子引力候选理论——NR 弦理论的最初表述中起到了关键作用 [2,3]。能否构建这样的 NR 弦理论，与“引力和量子力学结合是否需要狭义相对论”这一基础问题密切相关。此外，构建自洽的 NR 弦理论还引出了一个引人关注的问题: 量子引力的全息描述是否可以拓展为体区是 NR 引力、边界是 NR 共形场论的情形？除此以外，自文献 [4,5] 的开创性工作之后，NC 引力也在凝聚态物理中得到了诸多应用。在采用以几何方式实现 NR 对称性的有效场论描述时，该领域将引力场用作响应函数，以此获得对所研究模型非微扰性质的新认识。

Symmetries often help to resolve a problem in a given model or lead to useful restrictions on a model. A special kind of symmetry is supersymmetry. It is a space-time symmetry in the sense that the action of two supersymmetries leads to a space-time translation. Supersymmetry has been used as a possible symmetry of extensions of the standard model of particle physics. Finding evidence for supersymmetry is one of the research targets of the running LHC experiment. Supersymmetry is also an essential ingredient of superstring theory, where it solves the issue of how to avoid un-physical particles that would otherwise arise in the string spectrum. Assuming a small space-time curvature, the low-energy limit of superstring theory is described by supersymmetric extensions of general relativity called supergravity theories.

对称性通常有助于解决给定模型中的问题，或是对模型给出有用的约束。超对称是一类特殊的对称性，它是一种时空对称性: 两个超对称的作用结果是一个时空平移。超对称曾被认为是粒子物理标准模型拓展的可能对称性，寻找超对称证据是当前 LHC 实验的研究目标之一。超对称也是超弦理论的核心组成部分，它解决了弦谱中会出现非物理粒子的问题。假设时空曲率很小，超弦理论的低能极限由广义相对论的超对称拓展即超引力理论描述。

Non-relativistic supersymmetry is central to defining a NR version of superstring theory. The NR supersymmetry that arises in this context is much less well developed than relativistic supersymmetry (For early work on NR supersymmetry, see [6-9].). For instance, not much is known about NR superspace and the possible NR supermultiplets; these two concepts are central to the formulation of relativistic supersymmetry. Despite a huge amount of research on supergravity, thus far we do not have a supersymmetric extension of NC gravity that includes a supersym-metrization of Newton's law for gravity in four space-time dimensions. Presently known results about non-Lorentzian supergravity theories fall in two categories. On the one hand, three-dimensional (3*D*) supersymmetric NR supergravity theories have been constructed, whose underlying geometry is NC geometry with a co-dimension one distribution that splits the tangent space in a spatial subspace and a time direction. This class of theories includes a supersymmetric generalization of 3*D*NC gravity. On the other hand, recent work has also focused on ten-dimensional (10*D*) NR supergravity theories that employ a generalization of NC geometry, called stringy NC geometry, that is characterized by a co-dimension two distribution that splits the tangent space in an eight-dimensional 'transversal' subspace and two Minkowskian 'longitudinal' directions'. Whereas ordinary NC geometry naturally couples to NR particles, stringy NC geometry forms the natural background geometry for NR strings. Ten-dimensional supergravity theories based on stringy NC geometry can then be viewed as low energy approximations of NR string theory. It is the purpose of this review to give an overview of what is presently known about these 3*D* and 10*D* non-Lorentzian supergravity theories and give an outlook on, e.g., eleven-dimensional (11*D*) non-Lorentzian supergravity. Below, for the convenience of the reader, we give a short introduction to the recent literature.

非相对论超对称性是定义超弦理论非相对论版本的核心。该场景下得到的非相对论超对称性的发展程度远低于相对论超对称性 (关于非相对论超对称性的早期研究参见文献 [6-9])。例如，目前人们对非相对论超空间以及可能的非相对论超多重态了解有限，而这两个概念是相对论超对称性表述的核心。尽管针对超引力已有大量研究，迄今为止我们仍未得到非对易引力的超对称扩展，其中涵盖四维时空下牛顿引力定律的超对称化。目前已知的非洛伦兹超引力理论结果可分为两类。一方面，三维 (3*D*) 超对称非相对论超引力理论已被构建，其基础几何是非对易几何，该几何带有余维数为 1 的分布，可将切空间拆分为一个空间子空间和一个时间方向。这类理论包含 3*D*NC 引力的超对称推广。另一方面，近期研究也聚焦于十维 (10*D*) NR 超引力理论，这类理论采用了 NC 几何的推广，称为弦论非对易几何，其特征是带有余维数为 2 的分布，可将切空间拆分为一个八维“横向”子空间和两个闵可夫斯基“纵向”方向。普通非对易几何自然与非相对论粒子耦合，而弦论非对易几何则是非相对论弦的自然背景几何。基于弦论非对易几何的十维超引力理论可被视为非相对论弦论的低能近似。本综述的目的是概述目前关于这些 3*D* 和 10*D* 非洛伦兹超引力的已知研究成果，并对例如十一维 (11*D*) 非洛伦兹超引力等方向进行展望。下文为方便读者，我们将对近期相关文献做简要介绍。

For simplicity reasons, the first attempts to construct supersymmetric extensions of non-Lorentzian gravity have taken place in a 3*D* context. The advantage of working in 3*D* is not only that the calculations are simpler than in 4*D* but also that often gravity can be reformulated as a Chern-Simons theory that makes the relation with an underlying symmetry algebra very clear. Roughly speaking, one can distinguish between five kinds of efforts:

出于简化考虑，构建非洛伦兹引力超对称扩展的首批尝试是在 3*D* 背景下开展的。在 3*D* 中研究的优势不仅在于计算比 4*D* 中更简单，还在于引力通常可以重新表述为陈-西蒙斯理论，这使得它与基础对称代数的关系十分清晰。大致来说，相关研究工作可分为五类:

1. Gauging a supersymmetric extension of the Bargmann algebra, i.e., the centrally extended Galilei algebra

1. 对巴格曼代数 (即中心扩张伽利略代数) 的超对称扩展做规范处理

2. Taking a non-Lorentzian limit

2. 取非洛伦兹极限

3. Applying a NR superconformal tensor calculus

3. 应用非相对论超共形张量演算

4. Making use of the Chern-Simons formulation based on a non-Lorentzian super-algebra

4. 利用基于非洛伦兹超代数的陈-西蒙斯表述

5. Making Lie algebra expansions, either using Maurer-Cartan equations or semi-groups

5. 进行李代数展开，可采用毛雷尔-卡丹方程或半群方法

The first attempt to construct a 3*D*NC supergravity theory, i.e., a supersymmetric version of NC gravity, was undertaken in [10]. The construction was based on effort 1., i.e., the gauging of a $\mathcal{N} = 2$ supersymmetric extension of the Bargmann algebra. The reason that one considers $\mathcal{N} = 2$ supersymmetry is that any $\mathcal{N} = 1$ supersymmetric extension leads to a superalgebra, where the time translation or Hamilton operator cannot be written as the anti-commutator of two supersymmetries and therefore cannot be used for a physical realization of supersymmetry. The construction of [10] has two distinguishing features:

构建 3*D*NC 超引力理论 (即非对易引力的超对称版本) 的首次尝试由文献 [10] 完成。该构建基于上述第 1 类研究方向，即对巴格曼代数的 $\mathcal{N} = 2$ 超对称扩展做规范处理。人们考虑 $\mathcal{N} = 2$ 超对称性的原因是，任何 $\mathcal{N} = 1$ 超对称扩展得到的超代数中，时间平移算符即哈密顿算符无法表示为两个超对称算符的反对易子，因此无法用于超对称性的物理实现。文献 [10] 的构建有两个显著特点:

1. It is based on ordinary NC geometry that is appropriate to particles but not to strings. It can therefore only be identified with the compactification over a spatial longitudinal direction of a low-energy approximation of a non-Lorenzian superstring theory.

1. 它基于适用于粒子但不适用于弦的普通非对易几何，因此只能对应非洛伦兹超弦理论低能近似沿空间纵向方向的紧化结果。

2. By hand, a geometric constraint is imposed that implies that the underlying NC geometry admits a notion of absolute Newtonian time.

> 2. 研究人员手动施加了一个几何约束，该约束意味着基础非对易几何承认绝对牛顿时间的概念。

All transformation rules, including supersymmetry, and the expressions for all curvatures follow from the structure constants of the $\mathcal{N}=2$ super-Bargmann algebra. Next, the curvature tensors are divided into conventional curvatures - these are the curvatures that are set to zero in order to solve for the spin connections - and the remaining nonconventional curvatures. Setting one of the latter to zero corresponds to the geometric constraint that implies the presence of a notion of absolute time. A finite set of other (bosonic and fermionic) constraints is then obtained as follows. First, the supersymmetric variation of the absolute time constraint leads to two further constraints, one on a projection of the nonconventional gravitino curvature and one on the curvature of spatial rotations. At the same time, the commutator of two supersymmetries is considered, and this leads to two further fermionic constraints that can be considered as equations of motion for the gravitino field. This shows that we are dealing with an on-shell supergravity multiplet. Next, making use of the Bianchi identity for the boost curvature, one can show that the supersymmetric variation of the two fermionic constraints leads to a singlet component of the boost curvature that, after gauge fixing to rectilinearly accelerating reference frames, gives the Poisson equation for the Newton potential. There are no further constraints.

> 所有变换规则 (包括超对称) 以及所有曲率的表达式都可由 $\mathcal{N}=2$ 超巴格曼代数的结构常数导出。接下来，曲率张量被分为常规曲率与剩余非常规曲率: 常规曲率是为求解自旋联络而被置零的曲率。将其中一个非常规曲率置零对应于一个几何约束，该约束意味着存在绝对时间概念。随后，我们按如下步骤得到有限个其他 (玻色子与费米子) 约束: 首先，绝对时间约束的超对称变分导出另外两个约束，一个作用于非常规引力微子曲率的投影，另一个作用于空间旋转曲率。同时，通过研究两个超对称的对易子，可得到另外两个费米子约束，这两个约束可视为引力微子场的运动方程。这表明我们处理的是一个在壳超引力多重态。随后，利用 boost 曲率的比安基恒等式，可以证明两个费米子约束的超对称变分导出 boost 曲率的一个单态分量，在将规范固定为匀加速参考系后，该分量给出牛顿势的泊松方程。不存在其他约束。

The second attempt to construct a $\mathcal{N}=2$NC supergravity theory is based on effort 2., i.e., taking a non-Lorentzian limit of a $3D$ on-shell relativistic $\mathcal{N}=2$ supergravity theory consisting of a Dreibein and gravitino only [11] (See also references [12, 13].). It even leads, in a second step, to an off-shell version of such a non-Lorentzian supergravity theory. The construction of [11] uses a second-order formulation, where the spin connection fields have been solved by imposing conventional curvature constraints. It makes use of the same $\mathcal{N}=2$ super-Bargmann algebra as in [10], but this time it is shown how this non-Lorentzian superalgebra can be viewed as a special Inönü-Wigner contraction of a $\mathcal{N}=2$ Poincaré superalgebra with a central extension. In the non-Lorentzian limit procedure, one introduces a U(1) gauge field for this central extension, even though it is not part of the relativistic supergravity multiplet. In order not to upset the counting of the bosonic and fermionic degrees of freedom, its supercovariant curvature is set to zero by hand. The supersymmetric variation of this relativistic constraint leads to two more constraints (one fermionic and one bosonic) that can be considered as equations of motion.

构造 $\mathcal{N}=2$NC 超引力理论的第二种尝试基于方案 2，即取仅由三维标架和引力微子组成的 3D 在壳相对论性 $\mathcal{N}=2$ 超引力理论的非洛伦兹极限 [11](另见文献 [12,13])。该方案甚至在第二步进一步得到了这种非洛伦兹超引力理论的离壳版本。文献 [11] 的构造采用二阶表述，其中自旋联络场已通过施加常规曲率约束求解。该构造使用了与文献 [10] 相同的 $\mathcal{N}=2$ 超巴格曼代数，但本次工作证明了这种非洛伦兹超代数可被视为带中心扩张的 $\mathcal{N}=2$ 庞加莱超代数的特殊伊诺努-维格纳收缩。在非洛伦兹极限过程中，研究者为该中心扩张引入了一个 U(1) 规范场，尽管它并不属于相对论性超引力多重态。为了不打乱玻色子与费米子自由度计数，研究者手动将该规范场的超协变曲率置零。这个相对论性约束的超对称变分导出另外两个约束 (一个费米约束、一个玻色约束)，它们可被视为运动方程。

To define a non-Lorentzian limit of the Dreibein, the gravitino and the extra U(1) gauge field, a contraction is applied that is dual to the Inönü-Wigner contraction of the generators these gauge fields are associated to. The constraint that was imposed on the additional U(1) gauge field serves two purposes. First, substituting this constraint, for finite contraction parameter, into the expression for the dependent spin connection fields, makes it possible to define the non-Lorentzian limit of these spin connections without divergences. The non-Lorentzian limit of the supersymmetry rules is then non-divergent and reproduces the NR rules of [10]. Second, the non-Lorentzian limit of this constraint leads to the geometric constraint that endows the resulting NC geometry with an absolute time. Subsequent supersymmetric variations of this non-Lorentzian constraint, analysis of the supersymmetry algebra and making use of the NR Bianchi identities then show that one obtains the same non-Lorentzian NC supergravity theory as in [10] but this time by taking a non-Lorentzian limit.

为了定义三维标架、引力微子和额外 U(1) 规范场的非洛伦兹极限，研究者应用了一个与这些规范场所对应生成元的伊诺努-维格纳收缩对偶的收缩。施加在额外 U(1) 规范场上的约束有两个作用: 第一，对于有限收缩参数，将该约束代入依赖自旋联络场的表达式后，可以定义这些自旋联络的非发散非洛伦兹极限。此时超对称规则的非洛伦兹极限也不存在发散，并且重现了文献 [10] 的非相对论规则。第二，该约束的非洛伦兹极限导出几何约束，赋予最终得到的非交换几何绝对时间。随后对该非洛伦兹约束做超对称变分、分析超对称代数并利用非相对论比安基恒等式，结果表明: 我们通过非洛伦兹极限得到了与文献 [10] 相同的非洛伦兹非交换超引力理论。

The non-Lorentzian supergravity theories of [10] and [11] are based upon torsionless NC geometries with an absolute time constraint. As we will review in section ”Non-Lorentzian Geometry,” relaxing this constraint requires the introduction of a particular type of torsion, called intrinsic torsion. Newton-Cartan geometries with nonzero intrinsic torsion occur both in applications to condensed matter physics [14] and in applications to non-Lorentzian string theory [15]. The generalization of NC supergravity to include intrinsic torsion was undertaken in [16], by making use of effort 3., i.e., a NR superconformal tensor calculus that is based on a $\mathcal{N}=2$ Schrödinger superalgebra that contains dilatations and plays the role of the superconformal algebra in the relativistic case. The gauging of this Schrödinger superalgebra leads to an off-shell Schrödinger supergravity multiplet, which is the analogue of the relativistic Weyl multiplet. As was shown in [17], this gauging naturally leads to a Schrödinger supergravity multiplet with intrinsic torsion. The torsion is provided by the spatial components of the dilatation gauge field, which depend on the other fields of the multiplet. For this reason, the NR superconformal tensor calculus naturally leads to torsionful NC supergravity theories. Following the superconformal tensor calculus, the Schrödinger supergravity multiplet is coupled to two different choices of compensator matter multiplets [16]: a NR $3D\mathcal{N}=2$ scalar multiplet and a NR limit of a $\mathcal{N}=2$ vector multiplet. This leads to a so-called old minimal and new minimal off-shell $3D\mathcal{N}=2$NC supergravity multiplet.

文献 [10] 和 [11] 中的非洛伦兹超引力理论基于无挠非对易几何并满足绝对时间约束。正如我们将在“非洛伦兹几何”一节中回顾的，放松该约束需要引入一种特殊的挠，称为本征挠。具有非零本征挠的牛顿-嘉当几何既出现在凝聚态物理应用 [14] 中，也出现在非洛伦兹弦论应用 [15] 中。通过依托前述第 3 项工作，即基于包含膨胀变换的 $\mathcal{N} = 2$ 薛定谔超代数的非相对论超共形张量演算 (该超代数在相对论情形中起到超共形代数的作用)，文献 [16] 完成了对包含本征挠的非对易超引力的推广。对该薛定谔超代数的规范变换得到离壳薛定谔超引力多重态，它是相对论外尔多重态的对应物。正如文献 [17] 所示，这种规范变换自然得到带有本征挠的薛定谔超引力多重态，挠由膨胀规范场的空间分量提供，而该分量依赖于多重态的其他场。因此，非相对论超共形张量演算自然导出带挠非对易超引力理论。遵循超共形张量演算的框架，薛定谔超引力多重态可耦合两种不同选择的补偿物质多重态 [16]: 非相对论 $3D\mathcal{N} = 2$ 标量多重态和 $\mathcal{N} = 2$ 矢量多重态的非相对论极限，这就得到了所谓的旧最小离壳和新最小离壳 $3D\mathcal{N} = 2$NC 超引力多重态。

The fourth approach to $3D$ non-Lorentzian supergravity consists of constructing Chern-Simons formulations of gravity based on NR superalgebras with a nondegenerate bilinear form. The first example of such a Chern-Simons theory used an extension of the $\mathcal{N} = 2$ Bargmann superalgebra [18], whose bosonic part was considered in [19]. This superalgebra leads to an extension of NC supergravity called extended Bargmann supergravity. This is consistent with the fact that it is believed that there is no action for NC supergravity based upon the standard Bargmann algebra (For a proposal based upon a larger algebra, see [20].). The construction of an action for the extended theory is based upon the observation that, in order to write down an action for the single extra vector field that was added to general relativity in [11], one introduces two vector fields such that one can write down an action containing both vector fields. The underlying algebra is a Bargmann algebra with two central extensions instead of one. The supersymmetric extension of this extended Bargmann algebra was found by trial and error [18]. Remarkably, it contains beyond the expected two supersymmetry generators an extra fermionic generator.

$3D$ 非洛伦兹超引力的第四种研究方法是，基于带有非退化双线性型的非相对论超代数构造引力的陈-西蒙斯表述。这类陈-西蒙斯理论的首个例子使用了 $\mathcal{N} = 2$ 巴格曼超代数的扩张 [18]，其玻色子部分已在文献 [19] 中讨论。该超代数导出了扩张巴格曼超引力，它是非对易超引力的一种扩张。这与下述观点一致: 人们普遍认为基于标准巴格曼代数的非对易超引力不存在作用量 (关于基于更大代数的提案，见文献 [20])。扩张理论作用量的构造基于如下观察: 为了给文献 [11] 中添加到广义相对论的单个额外矢量场写下作用量，需要引入两个矢量场，从而可以构造出同时包含两个矢量场的作用量。其基础代数是带有两个中心扩张而非一个中心扩张的巴格曼代数。这种扩张巴格曼代数的超对称扩张是通过反复试错得到的 [18]。值得注意的是，除预期的两个超对称生成元外，它还包含一个额外的费米子生成元。

The paper [18] led to many follow-up papers that make use of the direct connection between a non-Lorentzian superalgebra (Non-Lorentzian superalgebras have been studied in [6-8] and [21-23].) with a non-degenerate bilinear form and a Chern-Simons formulation of a corresponding non-Lorentzian supergravity theory. For instance, in [24] a $3D$ supergravity version was constructed of the $4D$ non-Lorentzian gravity theory of [25] by using a certain extension of the Bargmann algebra that contains three additional generators. Further extended supergravity theories were constructed based on different extensions of the $\mathcal{N} = 2$ Bargmann superalgebra, such as extended Newton-Hooke supergravity, exotic Bargmann supergravity, extended Lifshitz supergravity, and extended Schrödinger supergravity [26]. More examples on non-Lorentzian Chern-Simons supergravity theories, including extensions with a cosmological constant and a related example in two dimensions, were given in [27-31].

文献 [18] 发表后催生了许多后续研究，这些工作利用了带有非退化双线性型的非洛伦兹超代数 (非洛伦兹超代数已在文献 [6-8] 和 [21-23] 中研究) 与对应非洛伦兹超引力理论的陈-西蒙斯表述之间的直接联系。例如，文献 [24] 通过使用包含三个额外生成元的巴格曼代数扩张，构造了文献 [25] 中 4D 非洛伦兹引力理论的 3D 超引力版本。基于 $\mathcal{N}$ = 巴格曼超代数的不同扩张，研究者还构造了更多扩张超引力理论，例如扩张牛顿-胡克超引力、奇异巴格曼超引力、扩张利夫希茨超引力和扩张薛定谔超引力 [26]。文献 [27-31] 给出了更多非洛伦兹陈-西蒙斯超引力理论的例子，包括带宇宙学常数的扩张和二维的相关例子。

Finally, a fifth method to construct a non-Lorentzian supergravity theory is by applying the so-called Lie algebra expansions. These expansions can be performed either by using Maurer-Cartan equations or semigroups. This method has been applied to re-derive the 3D extended Bargmann supergravity theory [33].

最后，构造非洛伦兹超引力理论的第五种方法是应用所谓的李代数展开。这类展开既可以通过莫勒-嘉当方程实现，也可以通过半群实现。该方法已被用于重新推导 3D 扩张巴格曼超引力理论 [33]。

Apart from 3D , this review also focuses on recent results in 10D . In particular, a non-Lorentzian version of 10D minimal supergravity, whose underlying structure is given by stringy NC geometry, has recently been constructed as a non-Lorentzian limit [34]. In this case, terms in the supersymmetry transformation rules of the fermions that diverge in the limit can be consistently eliminated by imposing particular geometric constraints. These constraints are invariant under the NR supersymmetry transformation rules that result from the limit. Applying the limit to the equations of motion of relativistic 10D minimal supergravity then leads to a set of NR equations of motion that consistently transform into each other under NR supersymmetry, once the geometric constraints are taken into account. After taking the limit, the theory is characterized by an emerging anisotropic scale symmetry, along with accompanying fermionic Stueckelberg symmetries. These symmetries imply that the NR supergravity multiplet that the limit leads to is shortened in comparison to its relativistic counterpart.

除 3D 外，本综述还重点介绍 10D 的最新研究成果。特别地，以弦论非对易几何为基础结构的 10D 极小超引力非洛伦兹版本，近期已作为非洛伦兹极限构造完成 [34]。在该情形下，费米子超对称变换规则中在此极限下发散的项，可以通过施加特定几何约束一致消去。这些约束在极限导出的非相对论超对称变换规则下保持不变。考虑几何约束后，对相对论性 10D 极小超引力的运动方程取该极限，会得到一组非相对论运动方程，它们在非相对论超对称下可 consistent 地相互变换。取极限后，该理论具有涌现的各向异性标度对称性，以及伴随的费米子斯图克尔贝格对称性。这些对称性表明，该极限得到的非相对论超引力多重态比其相对论对应更短。

This review is organized as follows. In section ”Lorentzian Geometry,” we first give a review of some basic notions in Lorentzian geometry to contrast them with a discussion of NC and stringy NC geometry in section ”Non-Lorentzian Geometry.” Next, in section ”3D Newton-Cartan Supergravity,” we discuss the construction and realization of $\mathcal{N} = 2$NC supergravity in three space-time dimensions. In particular, we show how, after partially gauge fixing its local symmetries, this gives rise to a supersymmetric version of Newtonian gravity. Subsequently, in section ”Non-Relativistic 3D Chern-Simons Supergravity and Lie Algebra Expansions”, we discuss NR 3D Chern-Simons supergravity theories. We focus on the theory of [18] that is based on an extension of the Bargmann algebra with three supercharges in particular. We also review how this algebra can be viewed as a Lie algebra expansion of the 3$D\mathcal{N} = 2$ super-Poincaré algebra [33]. In Section

5, we outline how a NR version of $10D$ minimal supergravity with an underlying stringy NC geometry arises as a non-Lorentzian limit of relativistic $10D$ minimal supergravity. Finally, in Section 6, we give a brief outlook on future developments.

本综述结构安排如下: 我们首先在“洛伦兹几何”一节回顾洛伦兹几何的若干基本概念，以便与“非洛伦兹几何”一节中对非对易几何和弦论非对易几何的讨论形成对比。接下来，我们在“三维牛顿-嘉当超引力”一节讨论三维时空中 $\mathcal{N} = 2$NC 超引力的构造与实现，我们尤其会说明，在对其局域对称性部分规范固定后，如何得到牛顿引力的超对称版本。随后，在“非相对论三维陈-西蒙斯超引力与李代数展开”一节，我们讨论非相对论 $3D$ 陈-西蒙斯超引力理论，重点介绍文献 [18] 提出的理论，该理论基于带有三个超荷的巴格曼代数扩张，我们还会回顾该代数如何可被视为 $3D\mathcal{N} = 2$ 超庞加莱代数的李代数展开 [33]。在第 5 节，我们概述以弦论非对易几何为基础的 $10D$ 极小超引力非相对论版本，如何作为相对论性 $10D$ 极小超引力的非洛伦兹极限得到。最后，我们在第 6 节对未来发展作简要展望。

Lorentzian and Non-Lorentzian Geometry

洛伦兹与非洛伦兹几何

In both relativistic and non-Lorentzian gravity, the gravitational interaction is modelled as the effect of matter moving in and curving the geometry of space-time. This geometry includes the specification of a metric and affine connection structure on an underlying space-time manifold and is described in a diffeomorphism covariant manner. The metric structure prescribes how to measure space-time distances, while the affine connection is used to define covariant derivatives of tensor fields and the ensuing notion of parallel transport along curves. In this section, we will review how these geometric ingredients are specified in non-Lorentzian geometry, in a manner that is adapted to their use in non-Lorentzian supergravity. To emphasize some important differences with Lorentzian geometry, we will first give a brief overview of aspects of the latter that are relevant for supergravity.

在相对论引力和非洛伦兹引力中，引力相互作用都被建模为物质在时空几何中运动并弯曲时空几何的效应。该几何需要指定底层时空流形上的度规与仿射联络结构，并以微分同胚协变的方式描述。度规结构规定了如何测量时空距离，而仿射联络则用于定义张量场的协变导数，以及由此得到的沿曲线平行移动的概念。在本节中，我们将梳理非洛伦兹几何中这些几何要素的指定方式，适配它们在非洛伦兹超引力中的应用。为了强调它和洛伦兹几何的一些重要区别，我们将首先简要概述后者与超引力相关的内容。

Lorentzian Geometry

洛伦兹几何

Relativistic gravity and supergravity use Lorentzian geometry, whose metric structure is given by a symmetric two-tensor field $g_{\mu\nu}$ (with the coordinate indices μ , $\upsilon, \cdots = 0, \ldots, D-1$, where D is the space-time dimension). This metric $g_{\mu\nu}$ is non-degenerate, i.e., constitutes an invertible matrix and one denotes its inverse by $g^{\mu\upsilon}$. The affine connection is a connection on the tangent bundle of the manifold and corresponds to a three-index field $\Gamma^{\rho}_{\mu\nu}$. In Lorentzian geometry, it satisfies the following metric compatibility condition:

相对论引力与超引力采用洛伦兹几何,其度量结构由对称二阶张量场 $g_{\mu\nu}$ 给出(坐标指标为 μ 、$\nu, \cdots = 0, \ldots, D-1$ ，其中 D 是时空维度)。该度量 $g_{\mu\nu}$ 是非退化的，即构成可逆矩阵，其逆矩阵记为 $g^{\mu\nu}$ 。仿射联络是流形切丛上的联络，对应一个三阶指标场 $\Gamma^{\rho}_{\mu\nu}$ 。在洛伦兹几何中，它满足如下度量相容性条件:

$$\partial_{\mu} g_{\nu\rho} - \Gamma^{\sigma}_{\mu\nu} g_{\sigma\rho} - \Gamma^{\sigma}_{\mu\rho} g_{\nu\sigma} = 0, \tag{1}$$

that ensures that the length of a vector does not change under parallel transport. Splitting $\Gamma^{\rho}_{\mu\nu}$ in a symmetric part $\Gamma^{\rho}_{(\mu\nu)}$ and an antisymmetric part $T^{\rho}_{\mu\nu} \equiv 2\Gamma^{\rho}_{[\mu\nu]}$, the $D^2(D+1)/2$ equations contained in (1) then express that the equal number of components of $\Gamma^{\rho}_{(\mu\nu)}$ is not independent but can instead be written in terms of the metric $g_{\mu\nu}$ and $T^{\rho}_{\mu\nu}$ as follows:

该条件保证平行移动过程中向量的长度不发生改变。将 $\Gamma^{\rho}_{\mu\nu}$ 分解为对称部分 $\Gamma^{\rho}_{(\mu\nu)}$ 和反对称部分 $T^{\rho}_{\mu\nu} \equiv 2\Gamma^{\rho}_{[\mu\nu]}$ ，式 (1) 包含的 $D^2(D+1)/2$ 个方程表明，$\Gamma^{\rho}_{(\mu\nu)}$ 的等数量分量并非独立，而可以通过度量 $g_{\mu\nu}$ 和 $T^{\rho}_{\mu\nu}$ 表示为如下形式:

$$\Gamma^{\rho}_{(\mu\nu)} = \frac{1}{2} g^{\rho\sigma} \left(2\partial_{(\mu} g_{\nu)\sigma} - \partial_{\sigma} g_{\mu\nu} \right) + g^{\rho\tau} T^{\sigma}_{\tau(\mu} g_{\nu)\sigma}. \tag{2}$$

The antisymmetric part $T^{\rho}_{\mu\nu}$ is called the torsion tensor, reflecting the fact that (unlike $\Gamma^{\rho}_{(\mu\nu)}$) it transforms tensorially under general coordinate transformations. Together with $g_{\mu\nu}$, $T^{\rho}_{\mu\nu}$ forms the independent data that define Lorentzian geometry. In general relativity, the torsion tensor is set to zero, and the corresponding affine connection is called the Levi-Civita connection. Relativistic supergravity by contrast contains nonzero torsion, constructed out of gravitino bilinears.

反对称部分 $T^{\rho}_{\mu\nu}$ 称为挠张量，这反映了一个性质: 它 (和 $\Gamma^{\rho}_{(\mu\nu)}$ 不同) 在一般坐标变换下按张量规律变换。挠张量与 $g_{\mu\nu}, T^{\rho}_{\mu\nu}$ 共同构成了定义洛伦兹几何的独立数据。在广义相对论中，挠张量被设定为零，对应的仿射联络称为列维-奇维塔联络。与之相对，相对论超引力包含非零挠，由引力子双线性构造得到。

While the affine connection suffices to couple gravity to bosonic matter fields that are sections of tensor products of the tangent bundle and its dual, it cannot be used to couple to sections of the spinor bundle, i.e., fermion fields. For this reason, in relativistic supergravity, one needs to use a different description of Lorentzian geometry that can accommodate fermions and that is called the Cartan or vielbein formulation. Its basic variable is the so-called vielbein one-form $E^{\hat{A}} = E_{\mu}{}^{\hat{A}}\, \mathrm{d}x^{\mu}$, where the index $\hat{A}$ can take the values $0, 1, \ldots, D-1$. Viewed as a matrix, $E_{\mu}{}^{\hat{A}}$ is assumed to be invertible, and its inverse is denoted by $E_{\hat{A}}{}^{\mu}$. The vielbein $E_{\mu}{}^{\hat{A}}$ corresponds to a square root of the metric:

仿射联络足以将引力耦合到切丛及其对偶张量积截面的玻色物质场，但无法用于耦合旋量丛的截面即费米场。因此，相对论超引力中需要使用一种能容纳费米子的不同洛伦兹几何描述，该描述称为嘉当表述或标架表述。其基本变量是所谓的标架一元形式 $E^{\hat{A}} = E_{\mu}{}^{\hat{A}}\, \mathrm{d}x^{\mu}$ ，其中指标 $\hat{A}$ 可取 $0, 1, \ldots, D-1$ 范围内的值。将其视为矩阵时，默认 $E_{\mu}{}^{\hat{A}}$ 可逆，其逆记为 $E_{\hat{A}}{}^{\mu}$ 。标架 $E_{\mu}{}^{\hat{A}}$ 对应度量的平方根:

$$g_{\mu\nu} = E_\mu{}^{\hat{A}} E_\nu{}^{\hat{B}} \eta_{\hat{A}\hat{B}}. \tag{3}$$

Vielbein that are related by the following infinitesimal action of local Lorentz transformations (with parameter $\Lambda^{\hat{A}\hat{B}} = -\Lambda^{\hat{B}\hat{A}}$)

满足局部洛伦兹变换无穷小作用关系的标架 (参数为 $\Lambda^{\hat{A}\hat{B}} = -\Lambda^{\hat{B}\hat{A}}$)

$$\delta E_\mu{}^{\hat{A}} = \Lambda^{\hat{A}}{}_{\hat{B}} E_\mu{}^{\hat{B}}, \tag{4}$$

gives rise to the same metric and should be seen as physically indistinguishable. The local Lorentz transformations (4) thus ought to be implemented as a gauge symmetry in the description of the geometry. To this end, the Cartan formulation not only introduces an affine connection $\Gamma^\rho_{\mu\nu}$ but also a gauge connection for local Lorentz transformations, i.e., a one-form field $\Omega_\mu{}^{\hat{A}\hat{B}} = -\Omega_\mu{}^{\hat{B}\hat{A}}$, that transforms as follows:

会给出相同的度量，应当被视为物理上不可区分。因此式 (4) 的局部洛伦兹变换必须作为几何描述中的规范对称性实现。为此，嘉当表述不仅引入了仿射联络 $\Gamma^\rho_{\mu\nu}$ ，还引入了局部洛伦兹变换的规范联络，即一元形式场 $\Omega_\mu{}^{\hat{A}\hat{B}} = -\Omega_\mu{}^{\hat{B}\hat{A}}$ ，其变换规律如下:

$$\delta\Omega_\mu{}^{\hat{A}\hat{B}} = \partial_\mu \Lambda^{\hat{A}\hat{B}} + 2\Lambda^{[\hat{A}|\hat{C}|]} \Omega_{\mu\hat{C}}{}^{\hat{B}]}. \tag{5}$$

This field $\Omega_\mu{}^{\hat{A}\hat{B}}$ is referred to as the ’ spin connection’ . Along with $\Gamma^\rho_{\mu\nu}$, it is constrained to obey the so-called vielbein postulate:

该场 $\Omega_\mu{}^{\hat{A}\hat{B}}$ 被称为“自旋联络”。它与 $\Gamma^\rho_{\mu\nu}$ 一同满足所谓的标架公设:

$$\partial_\mu E_\nu{}^{\hat{A}} - \Omega_\mu{}^{\hat{A}\hat{B}} E_{\nu\hat{B}} - \Gamma^\rho_{\mu\nu} E_\rho{}^{\hat{A}} = 0. \tag{6}$$

Using (3) and the antisymmetry of $\Omega_\mu{}^{\hat{A}\hat{B}}$ in the $\left[\hat{A}\hat{B}\right]$ indices, one sees that this postulate implies the metric compatibility condition (1) on the affine connection $\Gamma^\rho_{\mu\nu}$, so that the Cartan formulation reproduces the expression (2) for $\Gamma^\rho_{(\mu\nu)}$. Anti-symmetrizing (6) in the $[\mu\nu]$ indices leads to the following equation:

利用 (3) 式和 $\Omega_\mu{}^{\hat{A}\hat{B}}$ 在 $\left[\hat{A}\hat{B}\right]$ 指标下的反对称性，可看出该公设给出了仿射联络 $\Gamma^\rho_{\mu\nu}$ 满足的度规相容性条件 (1)，因此嘉当构造可以重现 $\Gamma^\rho_{(\mu\nu)}$ 的表达式 (2)。将 (6) 式在 $[\mu\nu]$ 指标下反对称化，得到以下方程:

$$R_{\mu\nu}{}^{\hat{A}}(P) \equiv 2\partial_{[\mu} E_{\nu]}{}^{\hat{A}} - 2\Omega_{[\mu}{}^{\hat{A}\hat{B}} E_{\nu]\hat{B}\hat{B}} = T^\rho_{\mu\nu} E_\rho{}^{\hat{A}}. \tag{7}$$

Viewing this as a set of $D^2(D-1)/2$ linear equations for the equal number of components of $\Omega_\mu{}^{\hat{A}\hat{B}}$, one finds that $\Omega_\mu{}^{\hat{A}\hat{B}}$ is not independent but is instead fully determined in terms of $E_\mu{}^{\hat{A}}$ and the torsion tensor:

将该式视为 $D^2(D-1)/2$ 个关于 $\Omega_\mu{}^{\hat{A}\hat{B}}$ 各分量的线性方程组，未知数的个数等于方程个数，可以发现 $\Omega_\mu{}^{\hat{A}\hat{B}}$ 并非独立场，而是完全由 $E_\mu{}^{\hat{A}}$ 和挠张量确定:

$$\Omega_{\mu}{}^{\hat{A}\hat{B}} = -2E^{[\hat{A}|\upsilon}\partial_{[\mu}E_{\upsilon]}{}^{\hat{B}]} + E^{\hat{A}\upsilon}E^{\hat{B}\rho}E_{\mu\hat{C}}\partial_{[\upsilon}E_{\rho]}{}^{\hat{C}} - \frac{1}{2}E^{\hat{A}\upsilon}E^{\hat{B}\rho}T^{\sigma}_{\upsilon\rho}g_{\sigma\mu}$$

$$+E^{[\hat{A}|\upsilon][E_{\rho}\hat{B}]}T^{\rho}_{\mu\upsilon}. \tag{8}$$

In this way, the Cartan formulation of Lorentzian geometry defines metric, affine and spin connection structures in terms of an independent vielbein and torsion tensor. The dependent spin connection (8) gives a connection on the spinor bundle that allows one to define covariant derivatives on fermionic fields and couple them to gravity. For instance, on an ordinary spinor field ψ , this covariant derivative is given by

由此，洛伦兹几何的嘉当构造以独立标架和挠张量为基础，定义了度规、仿射联络与自旋联络结构。由其他场确定的自旋联络 (8) 给出了旋量丛上的联络，从而可以定义费米场的协变导数，并将费米场与引力耦合。例如，对普通旋量场 ψ ，该协变导数可写为

$$D_{\mu}\psi = \partial_{\mu}\psi - \frac{1}{4}\Omega_{\mu}{}^{\hat{A}\hat{B}}\gamma_{\hat{A}\hat{B}}\psi. \tag{9}$$

In the supergravity literature, the Cartan formulation of Lorentzian geometry is often referred to as a ’gauging of the Poincaré algebra’ [35,36]. This terminology stems from the fact that the transformation rules (4), (5) are naturally interpreted as gauge transformation rules under Lorentz transformations, if one would naively gauge the Poincaré algebra. From this viewpoint, $E_{\mu}{}^{\hat{A}}$ and $\Omega_{\mu}{}^{\hat{A}\hat{B}}$ are interpreted as the components of a Poincaré algebra-valued gauge field along the translation and Lorentz transformation generators respectively. The two-form fields $R_{\mu\upsilon}{}^{\hat{A}}(P)$, defined in equation (7), are then similarly interpreted as the gauge-covariant curvature components along the translation generators. In the gauging language equation (7) is often referred to as a ’conventional constraint’. Conventional constraints are constraints on gauge-covariant curvature components that can be used to express a dependent field (such as the spin connection) in terms of other independent ones. They should be contrasted with what we will call geometric constraints that are constraints on the independent fields. Note that the term ’gauging of the Poincaré algebra’ should not be taken too literally, since $\Omega_{\mu}{}^{\hat{A}\hat{B}}$ is not an independent field, as would be the case in ordinary gauge theory. Nevertheless, thinking about Lorentzian geometry in gauge theoretic terms is useful for various generalizations, such as to supergravity and non-Lorentzian geometry, and we will frequently use this language in this review.

在超引力文献中，洛伦兹几何的嘉当构造常被称为“庞加莱代数规范化”[35,36]。该术语来源于: 如果对庞加莱代数做朴素规范化，变换规则 (4)(5) 可以自然地解释为洛伦兹变换下的规范变换规则。从该视角看，$E_{\mu}{}^{\hat{A}}$ 和 $\Omega_{\mu}{}^{\hat{A}\hat{B}}$ 分别被解释为取值于庞加莱代数的规范场，沿平移生成元和洛伦兹变换生成元的分量。方程 (7) 中定义的二形式场 $R_{\mu\upsilon}{}^{\hat{A}}(P)$ 同理可被解释为沿平移生成元的规范协变曲率分量。在规范理论的语言中，方程 (7) 常被称为“常规约束”。常规约束是对规范协变曲率分量的约束，可用来将非独立场 (如自旋联络) 表示为其他独立场的函数。与之相对的是我们所说的几何约束，这类约束是施加在独立场上的。注意“庞加莱代数规范化”这个说法不能从字面理解，因为 $\Omega_{\mu}{}^{\hat{A}\hat{B}}$ 并非普通规范理论中那样的独立场。尽管如此，从规范理论角度理解洛伦兹几何对各类推广 (如推广到超引力和非洛伦兹几何) 十分有用，我们在本综述中也会频繁使用该表述。

Non-Lorentzian Geometry

非洛伦兹几何

Non-Lorentzian geometry refers to differential geometric frameworks for space-times, whose local symmetry group differs from the Lorentz group. In this review, we will restrict ourselves to NR symmetry groups, which arise in or are extensions of $c \to \infty$ limits of the Lorentz group. The prime example of such a non-Lorentzian geometry is given by NC geometry [1, 37] that features local Galilean symmetries instead of the local Lorentz symmetries of Lorentzian geometry. Unlike Lorentzian geometry, NC geometry features two degenerate metrics. As a consequence, the structure of metric compatible connections (with or without torsion) also differs from that in Lorentzian geometry in crucial respects. In this section, we will first review (torsionful) NC geometry in section ”Newton-Cartan Geometry.” This is the relevant geometry for the $3D$ supergravity theories that we will discuss in sections ”3D Newton-Cartan Supergravity” and ”Non-Relativistic $3D$ Chern-Simons Supergravity and Lie Algebra Expansions.” The $10D$ supergravity theory of section ”10D Minimal Supergravity” on the other hand uses a recent generalization of NC geometry, called ’stringy Newton-Cartan geometry’, whose essential features will be reviewed in section ”Stringy Newton-Cartan Geometry.” This section is mostly based on [38] to which we refer for further details and references.

非洛伦兹几何指的是适用于时空的微分几何框架，其局部对称群不同于洛伦兹群。本综述中我们将讨论限制为 NR 对称群，这类对称群出现于洛伦兹群的 $c \to \infty$ 极限，或是对该极限的扩展。这类非洛伦兹几何的典型例子是 NC 几何 [1, 37]，它以局部伽利略对称性为特征，而非洛伦兹几何的局部洛伦兹对称性。与洛伦兹几何不同，NC 几何具有两个退化度量。因此，度量相容联络 (含或不含挠率) 的结构在关键方面也与洛伦兹几何中的结构不同。本节中，我们首先在“牛顿-嘉当几何”小节综述 (含挠率的)NC 几何，这是我们将在“三维牛顿-嘉当超引力”和“非相对论 $3D$ 陈-西蒙斯超引力与李代数扩张”小节讨论的 $3D$ 超引力理论对应的相关几何。另一方面，“10 维极小超引力”小节的 $10D$ 超引力理论使用了 NC 几何的一个最新推广，即“弦牛顿-嘉当几何”，我们会在“弦牛顿-嘉当几何”小节综述其核心特征。本节内容主要基于文献 [38]，更多细节和相关引用可参见该文献。

Newton-Cartan Geometry

牛顿-嘉当几何

A NC geometry is described by a D -dimensional differentiable manifold $\mathcal{M}$ (with local coordinates $x^\mu, \mu = 0, \ldots, D-1$) with particular degenerate metric and metric compatible connection structures. Here, we will review NC geometry in a frame field formulation that was developed in [39]. This formulation can be viewed as a gauging of the Bargmann algebra [40], i.e., the centrally extended Galilei algebra, and this is the language we will adopt here.

牛顿-嘉当 (NC) 几何由一个 D 维可微流形 $\mathcal{M}$ (配备局部坐标 $x^\mu, \mu = 0, \ldots, D-1$) 描述，具有特殊的退化度量和度量兼容联络结构。在此，我们将基于文献 [39] 发展的标架场表述回顾 NC 几何。该表述可看作巴格曼代数 (即中心扩张伽利略代数) 的规范化，我们将采用这一表述框架。

In D space-time dimensions, the generators of the Bargmann algebra consist of the time translation H , spatial translations P_a , Galilean boosts G_a , spatial rotations J_{ab} and the central charge M , where the spatial indices a, b assume values from 1 to $D-1$. The nontrivial commutation relations of the Bargmann algebra are given by:

在 D 维时空中，巴格曼代数的生成元包括时间平移 H 、空间平移 P_a 、伽利略 boost G_a 、空间转动 J_{ab} 和中心荷 M ，其中空间指标 a, b 的取值范围为 1 到 $D-1$ 。巴格曼代数的非对易关系如下:

$$[J_{ab}, P_c] = -2\delta_{c[a}P_{b]},\ [J_{ab}, G_c] = -2\delta_{c[a}G_{b]},\ [G_a, H] = -P_a,$$

$$[J_{ab}, J_{cd}] = 4\delta_{[a[c}J_{d]b]},\ [G_a, P_b] = -\delta_{ab}M. \tag{10}$$

Table 1 Summary of the gauge fields introduced in the gauging of the Bargmann algebra

表 1 巴格曼代数规范化中引入的规范场汇总

Symmetry	Generators	Gauge field
Time translation	H	τ_μ
Spatial translations	P_a	${e_\mu}^a$
Central charge	M	m_μ
Spatial rotations	J_{ab}	${\omega_\mu}^{ab}$
Galilean boosts	G_a	${\omega_\mu}^a$

The central charge M is physically interpretated as the Noether charge that corresponds to mass or particle number conservation in NR theories. Its inclusion in the gauging procedure is warranted if one wants to couple massive particles or fields to NC geometry.

中心荷 M 在物理上可解释为非相对论 (NR) 理论中对应质量或粒子数守恒的诺特荷。如果要将有质量粒子或场与 NC 几何耦合，就需要在规范化过程中引入该中心荷。

In the first step of the gauging procedure, one introduces a Bargmann algebra-valued gauge field, whose components along the various algebra generators are denoted by $\tau_\mu, {e_\mu}^a, m_\mu, {\omega_\mu}^{ab}$ and ${\omega_\mu}^a$ as outlined in Table 1. These fields transform as one-forms under general coordinate transformations. Their gauge transformation rules under local spatial rotations (with parameter λ^{ab}), local Galilean boosts (with parameter λ^a) and the local central charge transformation (with parameter σ) are determined by the structure constants of the Bargmann algebra and are given by:

规范化过程的第一步是引入一个巴格曼代数取值的规范场，其对应各代数生成元的分量记为 $\tau_\mu, {e_\mu}^a, m_\mu, {\omega_\mu}^{ab}$ 和 ${\omega_\mu}^a$ ，如表 1 所列。这些场在一般坐标变换下按一形式变换，它们在局域空间转动 (参数为 λ^{ab})、局域伽利略 boost(参数为 λ^a) 和局域中心荷变换 (参数为 σ) 下的规范变换规则由巴格曼代数的结构常数确定，具体为:

$$\delta\tau_\mu = 0,\ \delta{e_\mu}^a = \lambda^{ab}e_{\mu b} + \lambda^a\tau_\mu,\ \delta m_\mu = \partial_\mu\sigma + \lambda^a e_{\mu a}, \tag{11}$$

$$\delta{\omega_\mu}^{ab} = \partial_\mu\lambda^{ab} + 2\lambda^{[a|c|}{\omega_{\mu c}}^{b]},\ \delta{\omega_\mu}^a = \partial_\mu\lambda^a - {\omega_\mu}^{ab}\lambda_b + \lambda^{ab}\omega_{\mu b}. \tag{12}$$

Here and in the following, we have freely raised and lowered the flat spatial indices $a, b = 1, \ldots, D-1$ with Kronecker deltas δ^{ab}, δ_{ab} . For future reference, we also note that the field strengths of $\tau_\mu, e_\mu{}^a, m_\mu, \omega_\mu{}^{ab}$ and $\omega_\mu{}^a$ that are covariant with respect to the transformations of (11), (12) are given by:

下文中，我们统一用克罗内克 delta δ^{ab}, δ_{ab} 升降平坦空间指标 $a, b = 1, \ldots, D-1$ 。为方便后续讨论，我们给出相对于 (11)(12) 式变换协变的 $\tau_\mu, e_\mu{}^a, m_\mu, \omega_\mu{}^{ab}$ 和 $\omega_\mu{}^a$ 的场强:

$$R_{\mu\nu}(H) \equiv 2\partial_{[\mu}\tau_{\nu]}$$

$$R_{\mu\nu}{}^a(P) \equiv 2\partial_{[\mu}e_{\nu]}{}^a - 2\omega_{[\mu}{}^{ab}e_{\nu]b} - 2\omega_{[\mu}{}^a\tau_{\nu]},$$

$$R_{\mu\nu}(M) \equiv 2\partial_{[\mu}m_{\nu]} - 2\omega_{[\mu}{}^a e_{\nu]a},$$

$$R_{\mu\nu}{}^{ab}(J) \equiv 2\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^{[a|c|}\omega_{\nu]c}{}^{b]},$$

$$R_{\mu\nu}{}^a(G) \equiv 2\partial_{[\mu}\omega_{\nu]}{}^a - 2\omega_{[\mu}{}^{ab}\omega_{\nu]b}. \tag{13}$$

The field m_μ is the gauge field associated to the central charge M . In accordance to the physical interpretation of M , mentioned below (10), m_μ couples to conserved mass or particle number currents in theories of massive particles or fields in an arbitrary NC background. The one-form τ_μ is referred to as the time-like vielbein or clock form, whereas $e_\mu{}^a$ is referred to as the spatial Vielbein. Even though they do not constitute square invertible matrices, one can still define vectors τ^μ and $e_a{}^\mu$ that constitute 'inverse vielbein fields' in the sense that the following relations hold:

场 m_μ 是对应中心荷 M 的规范场。结合 (10) 式下方给出的 M 的物理解释，m_μ 可耦合任意 NC 背景下有质量粒子或场理论中的守恒质量流或守恒粒子数流。一形式 τ_μ 被称为类时标架场或时钟形式，$e_\mu{}^a$ 则被称为空间标架场。尽管它们不构成可逆方阵，仍可定义满足如下关系的矢量 τ^μ 和 $e_a{}^\mu$ 作为「逆标架场」:

$$\tau^\mu\tau_\mu = 1,\ \tau^\mu e_\mu{}^a = 0,\ \tau_\mu e_a{}^\mu = 0,$$

$$e_a{}^\mu e_\mu{}^b = \delta_a^b,\ \tau_\mu\tau^\nu + e_\mu{}^a e_a{}^\nu = \delta_\mu^\nu. \tag{14}$$

These inverse vielbein fields transform as follows under local spatial rotations and Galilean boosts:

这些逆标架场在局域空间转动和伽利略 boost 下的变换规则如下:

$$\delta\tau^\mu = -\lambda^a e_a{}^\mu,\ \delta e_a{}^\mu = \lambda_a{}^b e_b{}^\mu. \tag{15}$$

The metric structure of NC geometry is then defined in analogy to Lorentzian geometry by Galilean invariants that are quadratic in the vielbein or their inverses. Two such invariants can be found:

NC 几何的度量结构可以仿照洛伦兹几何，由标架场或其逆的二次型伽利略不变量定义。我们可以得到两类这样的不变量:

1. A covariant symmetric 2-tensor of rank 1, called the 'time-like metric'

1. 秩为 1 的协变对称二阶张量，称为「类时度量」

$$\tau_{\mu\upsilon} = \tau_\mu \tau_\upsilon \tag{16}$$

2. A contravariant symmetric 2-tensor of rank $D-1$, called the ' spatial (co-)metric'

2. 秩为 $D-1$ 的反变对称二阶张量，称为「空间 (余) 度量」

$$h^{\mu\upsilon} = e_a{}^\mu e_b{}^\upsilon \delta^{ab}. \tag{17}$$

These two metrics are mutually orthogonal in the sense that $h^{\mu\upsilon}\tau_{\upsilon\sigma} = 0$. They can be used to measure time-like and spatial distances along particular curves in the space-time. To see this, one first notes that one can use the time-like vielbein τ_μ to distinguish vectors into time-like and spatial ones. In particular, a vector X^μ is called time-like future (resp. past) directed whenever $\tau_\mu X^\mu > 0$ (resp. < 0) and spatial whenever $\tau_\mu X^\mu = 0$. A curve $\gamma : t \in [0,1] \mapsto x^\mu(t) \in \mathcal{M}$, whose tangent vectors $\dot{x}^\mu(t) \equiv \mathrm{d}x^\mu(t)/\mathrm{d}t$ are everywhere time-like future directed, can be regarded as the worldline of a physical observer that moves between two space-time points $x^\mu(0)$ and $x^\mu(1)$. The time interval Δt , needed by the observer to complete its motion between these two points, is then defined as

这两个度量在 $h^{\mu\upsilon}\tau_{\upsilon\sigma} = 0$ 的意义上相互正交。它们可用于测量时空特定曲线上的类时和空间距离。要理解这一点，首先需注意，可以使用类时标架 τ_μ 来区分向量是类时向量还是空间向量。特别地，当 $\tau_\mu X^\mu > 0$ (或 < 0) 时，向量 X^μ 被称为指向未来 (或过去) 的类时向量；当 $\tau_\mu X^\mu = 0$ 时，向量 X^μ 为空间向量。曲线 $\gamma : t \in [0,1] \mapsto x^\mu(t) \in \mathcal{M}$ 的切向量 $\dot{x}^\mu(t) \equiv \mathrm{d}x^\mu(t)/\mathrm{d}t$ 处处指向未来类时方向，可被视为在两个时空点 $x^\mu(0)$ 和 $x^\mu(1)$ 之间运动的物理观察者的世界线。观察者在这两点间完成运动所需的时间间隔 Δt 定义为

$$\Delta t \equiv \int_0^1 \mathrm{d}t \sqrt{\dot{x}^\mu \dot{x}^\upsilon \tau_{\mu\upsilon}} = \int_0^1 \mathrm{d}t \dot{x}^\mu \tau_\mu = \int_\gamma \mathrm{d}x^\mu \tau_\mu. \tag{18}$$

Likewise, the spatial distance ℓ along a curve $\sigma : s \in [0,1] \mapsto x^\mu(s) \in \mathcal{M}$, whose tangent vectors $x'^\mu(s) \equiv \mathrm{d}x^\mu(s)/\mathrm{d}s$ are everywhere spatial, is defined as

类似地，切矢量 $x'^\mu(s) \equiv \mathrm{d}x^\mu(s)/\mathrm{d}s$ 处处为空间矢量的曲线 $\sigma : s \in [0,1] \mapsto x^\mu(s) \in \mathcal{M}$ ，其空间距离 ℓ 定义为

$$\ell \equiv \int_0^1 \mathrm{d}s \sqrt{x'^\mu x'^\upsilon h_{\mu\upsilon}}, \text{ with } h_{\mu\upsilon} = e_\mu{}^a e_\upsilon{}^b \delta_{ab}. \tag{19}$$

Note that $h_{\mu\nu}$ satisfies

注意 $h_{\mu\nu}$ 满足

$$h^{\mu\upsilon}h_{\upsilon\rho} = \delta^{\mu}_{\rho} - \tau^{\mu}\tau_{\rho} \tag{20}$$

and thus is a right inverse of the spatial co-metric $h^{\mu\upsilon}$, when restricting its action to spatial vectors. This right inverse is however not boost invariant:

因此当将其作用限制在空间矢量上时，它是空间余度规 $h^{\mu\upsilon}$ 的右逆。但这个右逆不具有 boost 不变性:

$$\delta h_{\mu\upsilon} = 2\lambda^{a}\tau_{(\mu e_{\upsilon})a}. \tag{21}$$

Nevertheless, since the integral on the right-hand side of (19) is along a curve that satisfies $\tau_{\mu}x'^{\mu} = 0$, it is boost invariant.

尽管如此，由于 (19) 右侧的积分是在满足 $\tau_{\mu}x'^{\mu} = 0$ 的曲线上进行的，因此该积分具有 boost 不变性。

Having discussed the metric structure on a NC geometry, let us now turn to the definition of metric compatible connections. We have already introduced two one-form fields ${\omega_{\mu}}^{ab}$ and ${\omega_{\mu}}^{a}$, with gauge transformation rules (12), that can play the role of a NR analogue of the spin connection ${\Omega_{\mu}}^{\hat{A}\hat{B}}$. We will refer to ${\omega_{\mu}}^{ab}$ and ${\omega_{\mu}}^{a}$ as spin connections for spatial rotations and Galilean boosts, respectively. As in the Lorentzian case, these spin connections should not be independent but should instead depend on the other frame fields $\tau_{\mu}, {e_{\mu}}^{a}, m_{\mu}$ as well as torsion. This is achieved by requiring that the following conventional constraints, which are NR analogues of (7), hold identically:

讨论完牛顿-嘉当几何的度规结构后，我们现在转向适配度规的联络定义。我们已经引入了满足规范变换规则 (12) 的两个一形式场 ${\omega_{\mu}}^{ab}$ 和 ${\omega_{\mu}}^{a}$ ，它们可以充当自旋联络 ${\Omega_{\mu}}^{\hat{A}\hat{B}}$ 的非相对论类比。我们将 ${\omega_{\mu}}^{ab}$ 和 ${\omega_{\mu}}^{a}$ 分别称为空间转动自旋联络和伽利略 boost 自旋联络。与洛伦兹情况一样，这些自旋联络不应是独立的，而应当依赖于其他标架场 $\tau_{\mu}, {e_{\mu}}^{a}, m_{\mu}$ 和挠率。这可以通过要求下述常规约束恒成立来实现，这些约束就是式 (7) 的非相对论类比:

$$R_{\mu\upsilon}\left(P^{a}\right) \equiv 2\partial_{[\mu}{e_{\upsilon]}}^{a} - 2{\omega_{[\mu}}^{ab}e_{\upsilon]b} - 2{\omega_{[\mu}}^{a}\tau_{\upsilon]} = {T_{\mu\upsilon}}^{a},$$

$$R_{\mu\upsilon}\left(M\right) \equiv 2\partial_{[\mu}m_{\upsilon]} - 2{\omega_{[\mu}}^{a}e_{\upsilon]a} = T^{(m)}_{\mu\upsilon}. \tag{22}$$

Here, ${T_{\mu\upsilon}}^{a}$ and $T^{(m)}_{\mu\upsilon}$ are two torsion tensors that we will call the ' spatial torsion tensor' and 'mass torsion tensor', respectively. One can view (22) as a set of $D(D-1)^{2}/2 + D\left(D-1\right)/2 = D^{2}\left(D-1\right)/2$ linear algebraic equations for as many components of ${\omega_{\mu}}^{ab}$ and ${\omega_{\mu}}^{a}$. Solving these equations then yields the following expressions for the spin connections in terms of $\tau_{\mu}, {e_{\mu}}^{a}, m_{\mu}, {T_{\mu\upsilon}}^{a}$ and $T^{(m)}_{\mu\upsilon}$:

在此，${T_{\mu\upsilon}}^{a}$ 和 $T^{(m)}_{\mu\upsilon}$ 是两个挠张量，我们分别将其称为“空间挠张量”和“质量挠张量”。可以将式 (22) 看作对 ${\omega_{\mu}}^{ab}$ 和 ${\omega_{\mu}}^{a}$ 的对应分量建立的一组 $D(D-1)^{2}/2 + D\left(D-1\right)/2 = D^{2}\left(D-1\right)/2$ 线性代数方程。求解这些方程后即可得到自旋联络关于 $\tau_{\mu}, {e_{\mu}}^{a}, m_{\mu}, {T_{\mu\upsilon}}^{a}$ 和 $T^{(m)}_{\mu\upsilon}$ 的如下表达式:

$$\omega_\mu{}^a = \tau_\mu \tau^\nu e^{a\rho}\partial_{[\nu} m_{\rho]} + e^{a\nu}\partial_{[\mu} m_{\nu]} + e_{\mu b} e^{a\nu}\tau^\rho \partial_{[\nu} e_{\rho]}{}^b + \tau^\nu \partial_{[\mu} e_{\nu]}{}^a$$

$$-\tau_\mu \tau^\nu e^{a\rho} T^{(m)}_{\nu\rho} + e_{\mu b}\tau^\nu e^{(a|\rho|} T_{\nu\rho}{}^{b)} - \frac{1}{2} e_{\mu b} e^{b\nu} e^{a\rho} T^{(m)}_{\nu\rho},$$

$$\omega_\mu{}^{ab} = -2e^{[a|\nu|}\partial_{[\mu} e_{\nu]}{}^{b]} + e_{\mu c} e^{a\nu} e^{b\rho}\partial_{[\nu} e_{\rho]}{}^c - \tau_\mu e^{a\nu} e^{b\rho}\partial_{[\nu} m_{\rho]}$$

$$+\frac{1}{2}\tau_\mu e^{a\nu} e^{b\rho} T^{(m)}_{\nu\rho} + e^{[a|\nu|]} T_{\mu\nu}{}^{b]} - \frac{1}{2} e_{\mu c} e^{a\nu} e^{b\rho} T_{\nu\rho}{}^c. \tag{23}$$

Two comments are in order. First, note that the inclusion of the central charge of the Bargmann algebra in the gauging procedure is crucial to ensure that all spin connection components can be expressed in terms of other fields. Without the central charge and its associated gauge field m_μ , one would not be able to impose the second conventional constraint of (22) and the system of equations for the spin connection components would be underdetermined. One thus sees that, while m_μ does not play a role in defining the metric structure, it has a geometric significance as an ingredient that determines the connection structure of NC geometry.

这里有两点需要说明。首先，请注意，在规范过程中纳入 Bargmann 代数的中心荷对于保证所有自旋联络分量都可以用其他场表示至关重要。如果没有中心荷及其对应的规范场 m_μ ，就无法施加式 (22) 的第二个常规约束，自旋联络分量的方程组将是欠定的。由此可见，尽管 m_μ 在定义度量结构中不发挥作用，但它作为确定牛顿-卡坦几何联络结构的要素具有几何意义。

As a second comment, we remark that it is convenient to choose the torsion tensors $T_{\mu\nu}{}^a$ and $T^{(m)}_{\mu\nu}$ such that they transform under local spatial rotations and boosts as follows:

第二点说明，我们指出，选择满足如下局域空间转动和 boost 变换规律的挠张量 $T_{\mu\nu}{}^a$ 和 $T^{(m)}_{\mu\nu}$ 会更为方便:

$$\delta T_{\mu\nu}{}^a = \lambda^a{}_b T_{\mu\nu}{}^b + 2\lambda^a \partial_{[\mu}\tau_{\nu]}, \; \delta T^{(m)}_{\mu\nu} = \lambda_a T_{\mu\nu}{}^a. \tag{24}$$

This ensures that the local spatial rotation and boost transformations (induced by (11) and (24)) of the expressions (23) for the spin connections coincide with the rules (12) that are dictated by the Bargmann algebra. This can be checked either by direct calculation or by noting that the set of equations (22) is invariant under the spatial rotation and boost transformation rules of (11), (12) and (24). In the following, we will assume that (24) hold. The fact that $T_{\mu\nu}{}^a$ transforms under boosts to $\partial_{[\mu}\tau_{\nu]}$ indicates that the latter should also be interpreted as torsion. This will be confirmed in the following.

这可以保证, 自旋联络表达式 (23) 的局域空间转动和 boost 变换 (由式 (11) 和 (24) 诱导) 与 Bargmann 代数要求的规则 (12) 一致。这一点既可以通过直接计算验证，也可以通过注意到方程组 (22) 在式 (11)、(12) 和 (24) 的空间转动和 boost 变换下保持不变来验证。在下文中，我们假设式 (24) 成立。$T_{\mu\nu}{}^a$ 在 boost 变换下变换为 $\partial_{[\mu}\tau_{\nu]}$ 这一事实表明后者也应被解释为挠，我们接下来会对此加以确认。

Having discussed the spin connections $\omega_\mu{}^{ab}$ and $\omega_\mu{}^a$, one can define an affine connection $\Gamma^\rho_{\mu\nu}$ in analogy to the Lorentzian case, by imposing the following vielbein postulates:

讨论完自旋联络 $\omega_\mu{}^{ab}$ 和 $\omega_\mu{}^a$ 后，我们可以仿照洛伦兹的情况，通过施加如下标架公设定义仿射联络 $\Gamma^\rho_{\mu\nu}$：

$$\partial_\mu \tau_\upsilon - \Gamma^\rho_{\mu\upsilon}\tau_\rho = 0,\ \partial_\mu e_\upsilon{}^a - \omega_\mu{}^{ab} e_{\upsilon b} - \omega_\mu{}^a \tau_\upsilon - \Gamma^\rho_{\mu\upsilon} e_\rho{}^a = 0, \tag{25}$$

where $\omega_\mu{}^{ab}$ and $\omega_\mu{}^a$ are understood to be given by the expressions (23). It follows that $\Gamma^\rho_{\mu\nu}$ is compatible with the two metrics $\tau_{\mu\nu}$ and $h^{\mu\upsilon}$:

其中 $\omega_\mu{}^{ab}$ 和 $\omega_\mu{}^a$ 由式 (23) 给出。由此可得 $\Gamma^\rho_{\mu\nu}$ 与两个度量 $\tau_{\mu\nu}$ 和 $h^{\mu\upsilon}$ 相容:

$$\nabla_\mu \tau_{\upsilon\rho} \equiv \partial_\mu \tau_{\upsilon\rho} - \Gamma^\sigma_{\mu\upsilon}\tau_{\sigma\rho} - \Gamma^\sigma_{\mu\rho}\tau_{\upsilon\sigma} = 0$$

$$\nabla_\mu h^{\upsilon\rho} \equiv \partial_\mu h^{\upsilon\rho} + \Gamma^\upsilon_{\mu\sigma} h^{\sigma\rho} + \Gamma^\rho_{\mu\sigma} h^{\upsilon\sigma} = 0. \tag{26}$$

From (25) and (23), one finds that $\Gamma^\rho_{\mu\nu}$ can be written in terms of the NC metric structure, m_μ and the torsion tensors $T_{\mu\upsilon}{}^a, T^{(m)}_{\mu\upsilon}$ as follows:

由式 (25) 和 (23) 可知，$\Gamma^\rho_{\mu\nu}$ 可以通过牛顿-卡坦度量结构、m_μ 和挠张量 $T_{\mu\upsilon}{}^a, T^{(m)}_{\mu\upsilon}$ 表示为如下形式:

$$\Gamma^\rho_{\mu\upsilon} = \tau^\rho \partial_\mu \tau_\upsilon + \frac{1}{2} h^{\rho\sigma}\left(\partial_\mu h_{\sigma\upsilon} + \partial_\upsilon h_{\mu\sigma} - \partial_\sigma h_{\mu\upsilon}\right) + h^{\rho\sigma}\tau_\mu \partial_{[\sigma} m_{\upsilon]} + h^{\rho\sigma}\tau_\upsilon \partial_{[\sigma} m_{\mu]}$$

$$+ h^{\rho\sigma}\tau_{(\mu} T^m_{\upsilon)\sigma} - h^{\rho\sigma} e_{(\mu|a|} T_{\upsilon)\sigma}{}^a + \frac{1}{2} e_a{}^\rho T_{\mu\upsilon}{}^a. \tag{27}$$

One can explicitly check that this formula for $\Gamma^\rho_{\mu\nu}$ is invariant under local spatial rotations and boosts (even though boost invariance is not manifest) (Note that $\Gamma^\rho_{\mu\nu}$ will only be invariant under local central charge transformations, provided $T_{\mu\upsilon}{}^a$ and $T^{(m)}_{\mu\upsilon}$ are. In applications of NC geometry with torsion, e.g., in Lifshitz holography [15,41], one often encounters situations in which it is not possible to choose $T_{\mu\upsilon}{}^a$ and $T^{(m)}_{\mu\nu}$ such that they are simultaneously invariant under the central charge and transform as in (24) under local spatial rotations and boosts. In those cases, it is conventional/convenient to choose $T_{\mu\upsilon}{}^a$ and $T^{(m)}_{\mu\upsilon}$ with nontrivial central charge transformations, but such that (24) still hold, and to work with an affine connection $\Gamma^\rho_{\mu\nu}$ that is boost but not central charge invariant.). This is guaranteed by the fact that the dependent spin connections (23) transform as in (12).

我们可以明确验证，$\Gamma^\rho_{\mu\nu}$ 的这一公式在局域空间转动和 boost 下保持不变 (尽管 boost 不变性并非显而易见)(请注意，仅当 $T_{\mu\upsilon}{}^a$ 和 $T^{(m)}_{\mu\upsilon}$ 在局域中心荷变换下不变时，$\Gamma^\rho_{\mu\nu}$ 才会在该变换下不变。在带挠率的牛顿-卡坦几何的应用中，例如在李夫希茨全息学 [15,41] 中，我们常会遇到这类情况: 无法选取同时满足中心荷变换下不变、且在局域空间转动和 boost 下满足 (24) 式变换规律的 $T_{\mu\upsilon}{}^a$ 和 $T^{(m)}_{\mu\nu}$。在这类情况下，依惯例/方便起见，我们会选取满足非平凡中心荷变换、但仍满足 (24) 式的 $T_{\mu\upsilon}{}^a$ 和 $T^{(m)}_{\mu\upsilon}$，并使用满足 boost 不变性但不满足中心荷不变性的仿射联络 $\Gamma^\rho_{\mu\nu}$ 来进行计算。) 这一点可以由依赖自旋联络 (23) 满足 (12) 式变换规律这一事实保证。

The formula (27) gives the affine connection of NC geometry with arbitrary torsion. In particular, we see that its torsion $2\Gamma^\rho_{[\mu\nu]}$ is given by

公式 (27) 给出了带任意挠率的牛顿-卡坦几何的仿射联络。我们可以清楚看到，其挠率 $2\Gamma^{\rho}_{[\mu\nu]}$ 由下式给出

$$
2\Gamma^{\rho}_{[\mu\nu]} = 2\tau^{\rho}\partial_{[\mu}\tau_{\nu]} + e_a{}^{\rho}T_{\mu\nu}{}^{a}. \tag{28}
$$

The time-like component $2\Gamma^{\rho}_{[\mu\nu]}\tau_{\rho}$ is (as anticipated after (24)) given by $2\partial_{[\mu}\tau_{\nu]}$, while $T_{\mu\nu}{}^{a}$ gives the spatial components $2\Gamma^{\rho}_{[\mu\nu]}e_{\rho}{}^{a}$. The time-like component $2\partial_{[\mu}\tau_{\nu]}$ of the affine connection torsion is often called ’intrinsic torsion’ [42], since it does not appear in the dependent spin connections (23). Note that the notion of intrinsic torsion is absent in Lorentzian geometry, where the dependent spin connection (8) contains all components of $2\Gamma^{\rho}_{[\mu\nu]}$. A further difference between Lorentzian and NC geometry is that the latter features an extra mass torsion tensor $T^{(m)}_{\mu\nu}$ that appears in the spin connections (23) but not in the affine connection.

类时分量 $2\Gamma^{\rho}_{[\mu\nu]}\tau_{\rho}$ (正如我们在 (24) 式后预期的那样) 由 $2\partial_{[\mu}\tau_{\nu]}$ 给出, 而 $T_{\mu\nu}{}^{a}$ 给出空间分量 $2\Gamma^{\rho}_{[\mu\nu]}e_{\rho}{}^{a}$ 。仿射联络挠率的类时分量 $2\partial_{[\mu}\tau_{\nu]}$ 通常被称为“内禀挠率”[42]，因为它不出现在依赖自旋联络 (23) 中。请注意，洛伦兹几何中不存在内禀挠率的概念，在洛伦兹几何里，依赖自旋联络 (8) 已经包含了 $2\Gamma^{\rho}_{[\mu\nu]}$ 的所有分量。洛伦兹几何和牛顿-卡坦几何的另一个区别是，后者存在额外的质量挠率张量 $T^{(m)}_{\mu\nu}$ ，该张量出现在自旋联络 (23) 中，但不出现在仿射联络中。

It is interesting to consider situations in which the torsion is no longer arbitrary but instead some of its components vanish. For instance, since the intrinsic torsion components $2\Gamma^{\rho}_{[\mu\nu]}\tau_{\rho}$ are boost and rotation invariant, they can be consistently set to zero. This leads to the differential constraint

我们可以研究挠率不再任意、部分分量为零的情况，这很有意义。例如，由于内禀挠率分量 $2\Gamma^{\rho}_{[\mu\nu]}\tau_{\rho}$ 在 boost 和转动下不变，我们可以自洽地将其设为零，由此得到微分约束

$$
\partial_{[\mu}\tau_{\nu]} = 0 \tag{29}
$$

on the clock form. This illustrates another important difference with Lorentzian geometry, where truncating torsion tensor components does not give rise to constraints on the metric structure. The constraint (29) has a natural physical interpretation: it covariantly expresses the existence of an absolute time on the space-time. Indeed, if (29) holds, Stokes' theorem implies that two observers that move along different worldlines between the same initial and final space-time points, measure the same time interval (18) needed for their respective journeys. For modern applications of NC geometry, the absolute time constraint (29) is often too stringent and needs to be relaxed. This is, for instance, the case in Lifshitz holography, where gravity around so-called Lifshitz space-times is conjectured as a holographic dual of particular NR conformal field theories. The boundary of a Lifshitz space-time is described by a NC geometry, whose metric structure is only determined up to an anisotropic local Weyl rescaling with parameter Λ_D [15, 41] :

该约束作用在时钟形式上。这体现了牛顿-卡坦几何与洛伦兹几何的另一项重要区别: 截断挠率张量分量不会对洛伦兹几何的度规结构带来约束。约束 (29) 有自然的物理解释: 它协变地表述了时空中存在绝对时间。事实上，若 (29) 成立，斯托克斯定理表明，两个观测者在相同初末时空点之间沿不同世界线运动，测得的旅程时间间隔 (18) 是相同的。对于牛顿-卡坦几何的现代应用，绝对时间约束 (29) 通常过于严格，需要放宽。例如李夫希茨全息学就是这种情况: 在该理论中，所谓李夫希茨时空周围的引力被猜想为特定非相对论共形场论的全息对偶。李夫希茨时空的边界由牛顿-卡坦几何描述，该几何的度规结构仅在带参数 Λ_D [15, 41] 的各向异性局域外尔缩放范围内确定:

$$\delta\tau_\mu = z\Lambda_D\tau_\mu,\ \delta e_\mu{}^a = \Lambda_D e_\mu{}^a, \tag{30}$$

where z is a real number, called the dynamical exponent. The type of NC geometry that appears in Lifshitz holography cannot obey the constraint (29), as the latter is not invariant under the Weyl rescaling (30). A weaker constraint that is both boost and Weyl invariant and that can thus feature in Lifshitz holography is the so-called 'twistless torsional' constraint, in which only the spatial components (Similar to how curved indices are turned into flat ones in Lorentzian geometry, we will call the spatial components X_a/X^a of a one-form X_μ/ vector X^μ (and by extension of tensors) those that are obtained by contraction with $e_a{}^\mu/e_\mu{}^a$, according to the rule $X_a \equiv e_a{}^\mu X_\mu/X^a \equiv e_\mu{}^a X^\mu$. Likewise, the time-like components X_0/X^0 are defined as $X_0 \equiv \tau^\mu X_\mu/X^0 \equiv \tau_\mu X^\mu$. Since we allow the a -index to be raised and lowered with Kronecker deltas, the notation X^a/X_a can also stand for $e^{\mu a}X_\mu/e_{\mu a}X^\mu$, depending on the context. It is also convenient to raise and lower the 0 -index at the expense of a minus sign, so that X^0/X_0 can also be used to denote $-X_0 = -\tau^\mu X_\mu/-X^0 = -\tau_\mu X^\mu$, depending on the context.) of the intrinsic torsion are set to zero:

其中 z 为实数，称为动力学指数。利普希茨全息中出现的 NC 几何类型无法满足约束条件 (29)，因为该约束在魏尔缩放变换 (30) 下不具有不变性。可同时满足 boost 不变性和魏尔不变性、因此能够出现在利普希茨全息中的更弱约束，即所谓的“无扭转扭矩”约束，该约束仅要求内禀扭矩的空间分量为零:(类似在洛伦兹几何中将弯曲指标转换为平直指标的操作，我们将一维形式 X_μ/的向量 X^μ (可推广至张量) 的空间分量 X_a/X^a 定义为通过与 $e_a{}^\mu/e_\mu{}^a$ 缩并得到的分量，遵循规则 $X_a \equiv e_a{}^\mu X_\mu/X^a \equiv e_\mu{}^a X^\mu$。同理，类时分量 X_0/X^0 定义为 $X_0 \equiv \tau^\mu X_\mu/X^0 \equiv \tau_\mu X^\mu$。由于我们允许 a 指标通过克罗内克 delta 升降，因此根据上下文，符号 X^a/X_a 也可代表 $e^{\mu a}X_\mu/e_{\mu a}X^\mu$。此外，升降 0 指标时引入一个负号会更方便，因此根据上下文，X^0/X_0 也可用于表示 $-X_0 = -\tau^\mu X_\mu/-X^0 = -\tau_\mu X^\mu$。)

$$\tau_{[\mu}\partial_\nu\tau_{\rho]} = 0 \ \Leftrightarrow\ \tau_{ab} \equiv e_a{}^\mu e_b{}^\nu \partial_{[\mu}\tau_{\nu]} = 0. \tag{31}$$

A D -dimensional NC geometry with this twistless torsional constraint no longer exhibits a notion of absolute time. According to Frobenius' theorem, it can however still be foliated into $(D-1)$ -dimensional leaves that can be identified as spatial hypersurfaces of constant time.

满足该无扭转扭矩约束的 D 维 NC 几何不再具有绝对时间的概念。但根据弗罗贝尼乌斯定理，该几何仍可叶化为 $(D-1)$ 维叶，对应等时空间超曲面。

Stringy Newton-Cartan Geometry

弦型牛顿-嘉当几何

As we saw above, NC geometry is characterized by the existence of local frames (defined by τ_μ and ${e_\mu}^a$) that feature a time/space split and are related by local (homogeneous) Galilean transformations, as well as a one-form field m_μ that couples to conserved mass currents. As such, it is the geometrical arena in which the classical mechanics of NR point particles takes place. Recent years have witnessed a renewed interest in NR limits of extended objects in which the speed of light is taken to infinity only in the directions transverse to the objects under consideration. Applied to strings, such a limit leads to NR string theory [2,3,43], whose excitations satisfy NR dispersion relations and interact via NR gravity. The two-dimensional worldvolume of such NR strings is still relativistic and can therefore not naturally be embedded in NC geometry with a NR time/space split as described in the previous subsection. Recently, it has been shown that the natural background geometry in which NR strings move is given by a generalization of NC geometry that is called stringy Newton-Cartan geometry (see, e.g., [38,44-47] for an early example) and that we will review here.

正如我们上文所见，牛顿-嘉当 (NC) 几何的特征是存在由 τ_μ 和 ${e_\mu}^a$ 定义的局域标架: 这些标架满足时/空分解，由局域 (齐次) 伽利略变换联系，同时还存在一个耦合守恒质量流的一元场 m_μ 。它因此是非相对论 (NR) 质点经典力学的几何背景。近年来，延伸物体的非相对论极限重新引发了研究兴趣，这类极限仅令研究对象横截方向的光速趋于无穷。应用于弦时，这类极限得到非相对论弦论 [2,3,43]，该理论中弦的激发满足非相对论色散关系，并且通过非相对论引力相互作用。这类非相对论弦的二维世界面仍然是相对论性的，因此无法自然嵌入上一小节介绍的、带有非相对论时/空分解的牛顿-嘉当几何。近来研究表明，非相对论弦运动的自然背景几何是牛顿-嘉当几何的一种推广，称为弦型牛顿-嘉当几何 (相关早期例子见例如 [38,44-47])，我们将在此对其进行回顾。

Instead of a local time/space split, a D -dimensional stringy NC geometry features a split between two local so-called 'longitudinal' directions and the remaining $D-2$ 'transversal' ones. The longitudinal directions are equipped with a rank 2 Minkowski metric and can thus be used to embed the worldvolume of NR strings. A Cartan formulation of stringy NC geometry can be given in analogy to the one of NC geometry of the previous subsection. It is based on a 'longitudinal Vielbein' field ${\tau_\mu}^A$, with $A = 0, 1$, a ’ transversal Vielbein’ ${e_\mu}^a$, with $a = 2, \ldots, D-1$ and a two-form field $b_{\mu\upsilon}$. In what follows, we will freely raise and lower the A -index with a two-dimensional Minkowski metric $\eta_{AB} = \mathrm{diag}\,(-1, 1)$ and the a -index with a $(D-2)$ -dimensional Euclidean metric δ_{ab} .

弦型牛顿-嘉当几何不采用局域时/空分解，而是在 D 维弦型牛顿-嘉当几何中，将局域空间分解为两个所谓的“纵”方向和剩余的 $D-2$ 个“横”方向。纵方向配备了二阶闵氏度规，因此可用来嵌入非相对论弦的世界面。我们可以类比上一小节的牛顿-嘉当几何，给出弦型牛顿-嘉当几何的嘉当表述: 它基于“纵标架场” ${\tau_\mu}^A$ (满足 $A = 0, 1$)、“横标架” ${e_\mu}^a$ (满足 $a = 2, \ldots, D-1$)，以及一个二形式场 $b_{\mu\upsilon}$ 。在下文中，我们将用二维闵氏度规 $\eta_{AB} = \mathrm{diag}\,(-1, 1)$ 自由升降 A 指标，用 $(D-2)$ 维欧几里得度规 δ_{ab} 自由升降 a 指标。

The fields ${\tau_\mu}^A, {e_\mu}^a$ and $b_{\mu\upsilon}$ transform under local $(\mathrm{SO}\,(1,1) \times \mathrm{SO}\,(D-2)) \rtimes \mathbb{R}^{2(D-2)}$ transformations as follows:

场 $\tau_\mu{}^A, e_\mu{}^a$ 和 $b_{\mu\upsilon}$ 在局域 $(\mathrm{SO}(1,1)\times \mathrm{SO}(D-2))\rtimes\ \mathbb{R}^{2(D-2)}$ 变换下的变换规则如下:

$$\delta\tau_\mu{}^A = \lambda_M \varepsilon^A{}_B \tau_\mu{}^B,\ \delta e_\mu{}^a = \lambda^a{}_b e_\mu{}^b - \lambda_A{}^a \tau_\mu{}^A,$$

$$\delta b_{\mu\upsilon} = -2\varepsilon_{AB}\lambda^A{}_a \tau_{[\mu}{}^B e_{\upsilon]}{}^a. \tag{32}$$

Here, $\lambda_M, \lambda^{ab} = -\lambda^{ba}$ and λ^{Aa} are the parameters of $\mathrm{SO}(1,1)$, $\mathrm{SO}(8)$ and $\mathbb{R}^{2(D-2)}$, respectively. The $\mathrm{SO}(1,1)$ and $\mathrm{SO}(8)$ parts of this local symmetry group will be referred to as 'longitudinal Lorentz transformations' and 'transversal spatial rotations' . The $\mathbb{R}^{2(D-2)}$ part represents a type of boosts that transform transversal into longitudinal directions (but not vice versa) and that are called 'string Galilean boosts'. In addition to the string Galilean boost transformation (32), the two-form field $b_{\mu\upsilon}$ is also subjected to a one-form gauge symmetry, acting with parameter θ_μ as follows:

此处 $\lambda_M, \lambda^{ab} = -\lambda^{ba}$ 和 λ^{Aa} 分别是 $\mathrm{SO}(1,1)$, $\mathrm{SO}(8)$ 和 $\mathbb{R}^{2(D-2)}$ 的参数。这个局域对称群的 $\mathrm{SO}(1,1)$ 部分和 $\mathrm{SO}(8)$ 部分分别称为“纵洛伦兹变换”和“横空间转动”。$\mathbb{R}^{2(D-2)}$ 部分代表一类快度变换, 它将横方向变换为纵方向 (反之则不可), 这类变换称为“弦伽利略快度变换”。除了弦伽利略快度变换 (32), 二形式场 $b_{\mu\upsilon}$ 还满足一元规范对称性, 以参数 θ_μ 作用的变换规则如下:

$$\delta b_{\mu\nu} = 2\partial_{[\mu}\theta_{\nu]}. \tag{33}$$

As a consequence, $b_{\mu\nu}$ naturally couples to conserved string tension currents, in analogy to how the central charge gauge field m_μ of NC geometry couples to conserved mass currents [46].

因此, $b_{\mu\nu}$ 自然耦合守恒弦张力流, 这类比了牛顿-嘉当几何的中心荷规范场 m_μ 耦合守恒质量流的方式 [46]。

The longitudinal and transversal vielbein can be used to define the metric structure of stringy NC geometry. To do this, one first introduces 'inverse' longitudinal and spatial vielbein $\tau_A{}^\mu$ and $e_a{}^\mu$ that obey:

可利用纵向和横向标架定义弦牛顿-嘉当几何的度规结构。为此, 首先需要引入满足以下关系的“逆”纵向标架与空间标架 $\tau_A{}^\mu$ 和 $e_a{}^\mu$:

$$\tau_A{}^\mu \tau_\mu{}^B = \delta_A^B,\ \tau_A{}^\mu e_\mu{}^a = 0,\ e_a{}^\mu \tau_\mu{}^A = 0,$$

$$e_\mu{}^a e_b{}^\mu = \delta_b^a,\ \tau_\mu{}^A \tau_A{}^\upsilon + e_\mu{}^a e_a{}^\upsilon = \delta_\mu^\upsilon. \tag{34}$$

One can then construct the following two symmetric two-tensors that are quadratic in the (inverse) vielbein and that are invariant under the local $(\mathrm{SO}(1,1)\times \mathrm{SO}\ (D-2))\rtimes \mathbb{R}^{2(D-2)}$ transformations (32):

接下来可以构造如下两个对称二阶张量, 它们是 (逆) 标架的二次型, 且在局部 $(\mathrm{SO}(1,1)\times \mathrm{SO}\ (D-2))\rtimes \mathbb{R}^{2(D-2)}$ 变换 (32) 下保持不变:

$$\tau_{\mu\upsilon} = \tau_\mu{}^A \tau_\upsilon{}^B \eta_{AB},\ h^{\mu\upsilon} = e_a{}^\mu e_b{}^\upsilon \delta^{ab}. \tag{35}$$

The first of these has rank 2 and is called the longitudinal metric, whereas $h^{\mu\upsilon}$ has rank- $(D-2)$ and is called the transversal (co-)metric. Similar to NC geometry, the longitudinal metric can be used to determine the proper area of string worldsheets, while the transversal metric allows one to measure transversal distances to such worldsheets. We refer to [38] for more details on this.

第一个张量秩为 2，称为纵向度规；而 $h^{\mu\upsilon}$ 秩为- $(D-2)$，称为横向 (余) 度规。与普通牛顿-嘉当几何类似，纵向度规可用于确定弦世界面的固有面积，横向度规则可用于测量此类世界面的横向距离。更多细节我们参见文献 [38]。

The metric compatible connection of stringy NC geometry is determined by three spin connections $\omega_\mu, \omega_\mu{}^{ab}$ and $\omega_\mu{}^{Aa}$ that transform under longitudinal Lorentz transformations, transversal spatial rotations and string Galilean boosts as follows:

弦牛顿-嘉当几何的度规兼容联络由三个自旋联络 $\omega_\mu, \omega_\mu{}^{ab}$ 和 $\omega_\mu{}^{Aa}$ 确定，它们在纵向洛伦兹变换、横向空间转动和弦伽利略 boost 下的变换规则如下:

$$\delta\omega_\mu = \partial_\mu\lambda_M, \ \delta\omega_\mu{}^{ab} = \partial_\mu\lambda^{ab} + 2\lambda^{[a|c|}\omega_{\mu c}{}^{b]},$$

$$\delta\omega_\mu{}^{Aa} = \partial_\mu\lambda^{Aa} + \lambda_M\varepsilon^A{}_B\omega_\mu{}^{Ba} + \lambda^a{}_b\omega_\mu{}^{Ab} - \varepsilon^A{}_B\lambda^{Ba}\omega_\mu + \lambda^{Ab}\omega_{\mu b}{}^a. \tag{36}$$

These spin connections satisfy the following constraints in analogy to (22):

这些自旋联络满足类似于 (22) 的如下约束:

$$2\partial_{[\mu}\tau_{\upsilon]}{}^A - 2\varepsilon^A{}_B\omega_{[\mu}\tau_{\upsilon]}{}^B = T_{\mu\upsilon}{}^A$$

$$2\partial_{[\mu}e_{\upsilon]}{}^a - 2\omega_{[\mu}{}^{ab}e_{\upsilon]b} + 2\omega_{[\mu}{}^{Aa}\tau_{\upsilon]A} = T_{\mu\upsilon}{}^a,$$

$$3\partial_{[\mu}b_{\upsilon\rho]} + 6\varepsilon_{AB}\omega_{[\mu}{}^{Ab}\tau_\upsilon{}^B e_{\rho]b} = T^{(b)}_{\mu\upsilon\rho}, \tag{37}$$

where the left-hand sides are covariant with respect to (36) and $T_{\mu\nu}{}^A, T_{\mu\nu}{}^a, T^{(b)}_{\mu\nu\rho}$ are suitable torsion tensors. Not all of the constraints (37) are conventional, since not all of them contain the spin connections $\omega_\mu, \omega_\mu{}^{ab}$ and $\omega_\mu{}^{Aa}$. In particular, contracting (37) with the inverse vielbein $\tau_A{}^\mu, e_a{}^\mu$ and taking suitable (anti-)symmetrizations, one finds that (37) contains the following components:

其中左侧相对于 (36) 是协变的，$T_{\mu\nu}{}^A, T_{\mu\nu}{}^a, T^{(b)}_{\mu\nu\rho}$ 是合适的挠张量。并非所有约束 (37) 都是常规的，因为并非所有约束都包含自旋联络 $\omega_\mu, \omega_\mu{}^{ab}$ 和 $\omega_\mu{}^{Aa}$ 。具体来说，将逆标架 $\tau_A{}^\mu, e_a{}^\mu$ 缩并到 (37) 并做适当的 (反) 对称化后，可发现 (37) 包含以下分量:

$$2\tau_{(A|}{}^\mu e_a{}^\upsilon\partial_{[\mu}\tau_{\upsilon]|B)} = \tau_{(A|}{}^\mu e_a{}^\upsilon T_{\mu\upsilon|B)}, \ 2e_a{}^\mu e_b{}^\upsilon\partial_{[\mu}\tau_{\upsilon]}{}^A = e_a{}^\mu e_b{}^\upsilon T_{\mu\upsilon}{}^A,$$

$$3e_a{}^\mu e_b{}^\upsilon e_c{}^\rho\partial_{[\mu}b_{\upsilon\rho]} = e_a{}^\mu e_b{}^\upsilon e_c{}^\rho T^{(b)}_{\mu\upsilon\rho}, \tag{38}$$

that are independent of the spin connections. This leaves $D+D(D-1)(D-2)/2+\ (D-2)^2$ equations of (37) that can be used to solve and express the spin connections in terms of $\tau_\mu{}^A, e_\mu{}^a, b_{\mu\nu}, T_{\mu\nu}{}^A, T_{\mu\nu}{}^a$ and $T^{(b)}_{\mu\nu\rho}$. Since $\omega_\mu, \omega_\mu{}^{ab}$ and $\omega_\mu{}^{Aa}$ have a total of $D+D(D-2)(D-3)/2+2D(D-2)$ components, one sees that not all spin connection components can be expressed in this way. In particular, the $2(D-2)$ components

这些分量与自旋联络无关。这就使得 (37) 中还剩余 $D+D(D-1)(D-2)/2+\ (D-2)^2$ 个方程，可用于求解并将自旋联络表示为 $\tau_\mu{}^A, e_\mu{}^a, b_{\mu\nu}, T_{\mu\nu}{}^A, T_{\mu\nu}{}^a$ 和 $T^{(b)}_{\mu\nu\rho}$ 的形式。由于 $\omega_\mu, \omega_\mu{}^{ab}$ 和 $\omega_\mu{}^{Aa}$ 总共有 $D+D(D-2)(D-3)/2+2D(D-2)$ 个分量，可知并非所有自旋联络分量都可以用这种方式表示。具体而言，$2(D-2)$ 分量

$$\tau_{\{A|}{}^\mu\omega_{\mu|B\}}{}^a \equiv \tau_{(A|}{}^\mu\omega_{\mu|B)}{}^a - \frac{1}{2}\eta_{AB}\tau_C{}^\mu\omega_\mu{}^{Ca}, \tag{39}$$

remain as independent fields in the connection structure of stringy NC geometry. All other spin connection components can be expressed in terms of the longitudinal and transversal Vielbeine, $b_{\mu\nu}$ and the torsion tensors appearing in (37). We refer to [38] for their explicit expressions and for details on how these can be obtained.

在弦牛顿-嘉当几何的联络结构中始终是独立场。所有其他自旋联络分量都可以用纵向和横向标架、$b_{\mu\nu}$ 以及 (37) 中出现的挠张量表示。它们的显式表达式以及推导细节我们参见文献 [38]。

Once the spin connections $\omega_\mu, \omega_\mu{}^{ab}$ and $\omega_\mu{}^{Aa}$ have been expressed in this way, an affine connection $\Gamma^\rho_{\mu\nu}$ can be introduced via the following vielbein postulates:

当自旋联络 $\omega_\mu, \omega_\mu{}^{ab}$ 和 $\omega_\mu{}^{Aa}$ 以这种方式表示后，就可以通过如下标架假设引入仿射联络 $\Gamma^\rho_{\mu\nu}$：

$$\partial_\mu\tau_\upsilon{}^A - \varepsilon^A{}_B\omega_\mu\tau_\upsilon{}^B - \Gamma^\rho_{\mu\upsilon}\tau_\rho{}^A = 0$$

$$\partial_\mu e_\upsilon{}^a - \omega_\mu{}^{ab}e_{\upsilon b} + \omega_\mu{}^{Aa}\tau_{\upsilon A} - \Gamma^\rho_{\mu\upsilon}e_\rho{}^a = 0. \tag{40}$$

This connection $\Gamma^\rho_{\mu\nu}$ is by construction compatible with the metrics $\tau_{\mu\nu}$ and $h^{\mu\nu}$ of (35)

根据构造，该联络 $\Gamma^\rho_{\mu\nu}$ 与式 (35) 中的度规 $\tau_{\mu\nu}$ 和 $h^{\mu\nu}$ 相容

$$\nabla_\mu\tau_{\upsilon\rho} \equiv \partial_\mu\tau_{\upsilon\rho} - \Gamma^\sigma_{\mu\upsilon}\tau_{\sigma\rho} - \Gamma^\sigma_{\mu\rho}\tau_{\upsilon\sigma} = 0$$

$$\nabla_\mu h^{\upsilon\rho} \equiv \partial_\mu h^{\upsilon\rho} + \Gamma^\upsilon_{\mu\sigma}h^{\sigma\rho} + \Gamma^\rho_{\mu\sigma}h^{\upsilon\sigma} = 0 \tag{41}$$

In analogy to the NC case, one can ensure that $\Gamma^\rho_{\mu\nu}$ is invariant under local longitudinal Lorentz transformations, transversal spatial rotations and Galilean boosts, by choosing the transformation rules of the tensors $T_{\mu\upsilon}{}^A, T_{\mu\upsilon}{}^a$ and $T^{(b)}_{\mu\upsilon\rho}$ as follows:

与牛顿-卡坦 (NC) 情形类似，我们可以通过如下方式选定张量 $T_{\mu\upsilon}{}^A, T_{\mu\upsilon}{}^a$ 和 $T^{(b)}_{\mu\upsilon\rho}$ 的变换规则，保证 $\Gamma^\rho_{\mu\nu}$ 在局部纵向洛伦兹变换、横向空间转动和伽利略 boost 下保持不变:

$$\delta T_{\mu\nu}{}^{A} = \lambda_M \varepsilon^{A}{}_{B} T_{\mu\nu}{}^{B},\ \delta T_{\mu\nu}{}^{a} = \lambda^{a}{}_{b} T^{b}_{\mu\nu} - \lambda_{A}{}^{a} T_{\mu\nu}{}^{A},$$

$$\delta T^{(b)}_{\mu\nu\rho} = -3\varepsilon_{AB}\lambda^{A}{}_{a} T_{[\mu\nu}{}^{B} e_{\rho]}{}^{a} + 3\varepsilon_{AB}\lambda^{A}{}_{a} T_{[\mu\nu}{}^{a}\tau_{\rho]}{}^{B}. \tag{42}$$

From (40), one can solve and express $\Gamma^{\rho}_{\mu\nu}$ in terms of the stringy NC metrics (35), $b_{\mu\nu}$ and the tensors $T_{\mu\nu}{}^{A}, T_{\mu\nu}{}^{a}$ and $T^{(b)}_{\mu\nu\rho}$ (see [38] for an explicit expression).

根据式 (40)，我们可以求解并将 $\Gamma^{\rho}_{\mu\nu}$ 用弦论牛顿-卡坦度规 (35)、$b_{\mu\nu}$ 以及张量 $T_{\mu\nu}{}^{A}, T_{\mu\nu}{}^{a}$ 和 $T^{(b)}_{\mu\nu\rho}$ 表示 (显式表达式可参见文献 [38])。

The metric compatible affine connection $\Gamma^{\rho}_{\mu\nu}$ introduced above features generic torsion, given by

上文引入的与度规相容的仿射联络 $\Gamma^{\rho}_{\mu\nu}$ 具有一般形式的挠率，挠率由下式给出

$$2\Gamma^{\rho}_{[\mu\nu]} = \tau_{A}{}^{\rho} T_{\mu\nu}{}^{A} + e_{a}{}^{\rho} T_{\mu\nu}{}^{a}. \tag{43}$$

As in the NC case, part of this torsion is intrinsic. In particular, the torsion components $T_{(A|a|B)} \equiv \tau_{(A|}{}^{\mu} e_{a}{}^{\upsilon} T_{\mu\upsilon|B)}$ and $T_{ab}{}^{A} \equiv e_{a}{}^{\mu} e_{b}{}^{\upsilon} T_{\mu\upsilon}{}^{A}$ (We follow a similar convention as in footnote 8 to turn curved indices into longitudinal or transversal ones. In particular, we will call the longitudinal components X_A/X^A of a one-form X_μ /vector X^μ (and by extension of tensors) those that are obtained by contraction with $\tau_A{}^{\mu}/\tau_\mu{}^{A}$, according to the rule $X_A \equiv \tau_A{}^{\mu} X_\mu / X^A \equiv \tau_\mu{}^{A} X^\mu$. Likewise, the transversal components X_a/X^a are defined as $X_a \equiv e_a{}^{\mu} X_\mu / X^a \equiv\ e_\mu{}^{a} X^\mu$.) do not appear in the dependent spin connections and constitute the intrinsic torsion of the stringy NC affine connection. From (38), one sees that setting this intrinsic torsion equal to zero leads to differential constraints on the longitudinal vielbein $\tau_\mu{}^{A}$, similar to what happens in the NC case. As in the NC case, one can consistently set intrinsic torsion tensor components equal to zero in various ways. We refer to [38] for an in-depth analysis of the different possibilities and their geometrical interpretation.

和 NC 情形一样, 该挠率的一部分是内禀的。具体而言, 挠率分量 $T_{(A|a|B)} \equiv \tau_{(A|}{}^{\mu} e_{a}{}^{\upsilon} T_{\mu\upsilon|B)}$ 和 $T_{ab}{}^{A} \equiv e_{a}{}^{\mu} e_{b}{}^{\upsilon} T_{\mu\upsilon}{}^{A}$ (我们遵循脚注 8 的类似约定，将弯曲指标转换为纵向或横向指标。具体来说，对于一形式 X_μ /矢量 X^μ ，(以及推广到张量的情形) 其纵向分量 X_A/X^A 是指通过如下规则与 $\tau_A{}^{\mu}/\tau_\mu{}^{A}$ 缩并得到的分量: $X_A \equiv \tau_A{}^{\mu} X_\mu / X^A \equiv \tau_\mu{}^{A} X^\mu$ 。类似地，横向分量 X_a/X^a 按 $X_a \equiv e_a{}^{\mu} X_\mu / X^a \equiv\ e_\mu{}^{a} X^\mu$.) 定义) 不会出现在依赖的自旋联络中，构成弦论牛顿-卡坦仿射联络的内禀挠率。从式 (38) 可以看出，令该内禀挠率为零会对纵向标架 $\tau_\mu{}^{A}$ 给出微分约束，这与 NC 情形的结果类似。和 NC 情形一样，我们可以通过多种方式一致地将内禀挠率张量分量置零。关于不同可能性及其几何解释的深入分析，请参见文献 [38]。

Having reviewed the basics of Lorentzian and non-Lorentzian geometry, we are now ready to discuss different non-Lorentzian supergravity theories. We start by discussing in the next section 3*D* Newton-Cartan supergravity.

回顾完洛伦兹几何与非洛伦兹几何的基础内容后, 我们现在可以着手讨论不同的非洛伦兹超引力理论了。我们在下一节 3*D* 首先讨论牛顿-卡坦超引力。

3D Newton-Cartan Supergravity

三维 Newton-Cartan 超引力

In this section, we will review the $3D$ supersymmetric version of NC gravity of [10,11]. Originally, this theory was constructed as a gauging of a $\mathcal{N} = 2$ super-Bargmann algebra [10]. In [11], it was rederived by applying a non-Lorentzian limit procedure to relativistic $3D, \mathcal{N} = 2$ supergravity. Since recent efforts to construct non-Lorentzian supergravity theories in higher dimensions are mainly based on taking a non-Lorentzian limit, we will here focus in detail on the construction of [11], in order to allow for comparison with the $10D$ non-Lorentzian supergravity of section "10D Minimal Supergravity." In section "3D $\mathcal{N} = 2$ On-shell Newton-Cartan Supergravity," we will first review the construction of $3D$ NC supergravity via a non-Lorentzian limit. In section "Gauge Fixing," we will then show how a supersymmetric version of Newtonian gravity can be obtained as a gauge fixing of NC supergravity.

本节我们将回顾现有文献中 $3D$ 版本的 NC 超对称引力。该理论最初是通过对 $\mathcal{N} = 2$ 超 Bargmann 代数进行规范构造得到的 [10]。文献 [11] 通过对相对论性 $3D, \mathcal{N} = 2$ 超引力取非洛伦兹极限，重新推导出了该理论。由于近来构造高维非洛伦兹超引力理论的工作大多基于取非洛伦兹极限的方法，本文将详细介绍文献 [11] 的构造过程，以便与“10 维最小超引力”一节中的 $10D$ 非洛伦兹超引力进行对比。在“三维 $\mathcal{N} = 2$ 壳上 Newton-Cartan 超引力”一节中，我们首先回顾通过非洛伦兹极限构造 $3D$ NC 超引力的过程。随后在“规范固定”一节中，我们将说明如何通过对 NC 超引力做规范固定得到牛顿引力的超对称版本。

3D $\mathcal{N} = 2$ On-shell Newton-Cartan Supergravity

三维 $\mathcal{N} = 2$ 壳上牛顿-嘉当超引力

Our starting point is the relativistic $3D\mathcal{N} = 2$ supergravity multiplet with field content $\left\{E_\mu{}^{\hat{A}}, \Psi_{\mu i}\right\}\left(\hat{A} = 0, 1, 2; i = 1, 2\right)$. Under diffeomorphisms (with parameter ξ^μ) and Lorentz rotations (with parameter $\Lambda^{\hat{A}\hat{B}}$), these fields transform as

我们的出发点是场内容为 $\left\{E_\mu{}^{\hat{A}}, \Psi_{\mu i}\right\}\left(\hat{A} = 0, 1, 2; i = 1, 2\right)$ 的相对论性 $3D\mathcal{N} = 2$ 超引力多重态。在微分同胚 (参数为 ξ^μ) 和洛伦兹转动 (参数为 $\Lambda^{\hat{A}\hat{B}}$) 下，这些场的变换为

$$\delta E_\mu{}^{\hat{A}} = \xi^\upsilon \partial_\upsilon E_\mu{}^{\hat{A}} + \partial_\mu \xi^\upsilon E_\upsilon{}^{\hat{A}} + \Lambda^{\hat{A}}{}_{\hat{B}} E_\mu{}^{\hat{B}}, \tag{44}$$

$$\delta \Psi_{\mu i} = \xi^\upsilon \partial_\upsilon \Psi_{\mu i} + \partial_\mu \xi^\upsilon \Psi_{\upsilon i} + \frac{1}{4} \Lambda^{\hat{A}\hat{B}} \gamma_{\hat{A}\hat{B}} \Psi_{\mu i}. \tag{45}$$

Furthermore, the supersymmetry transformation rules (with parameter η_i) are given by

此外，超对称变换规则 (参数为 η_i) 由下式给出

$$\delta E_\mu{}^{\hat{A}} = \frac{1}{2} \delta^{ij} \overline{\eta}_i \gamma^{\hat{A}} \Psi_{\mu j}, \tag{46}$$

$$\delta\Psi_{\mu i} = D_\mu \eta_i = \partial_\mu \eta_i - \frac{1}{4}\Omega_\mu{}^{\hat{A}\hat{B}}(E, \Psi_i)\gamma_{\hat{A}\hat{B}}\eta_i, \tag{47}$$

where D_μ is the Lorentz-covariant derivative and the dependent spin connection $\Omega_\mu{}^{\hat{A}\hat{B}}(E, \Psi_i)$ is given by

其中 D_μ 是洛伦兹协变导数，依赖型自旋联络 $\Omega_\mu{}^{\hat{A}\hat{B}}(E, \Psi_i)$ 由下式给出

$$\Omega_\mu{}^{\hat{A}\hat{B}}(E, \Psi_i) = -2E^{\nu[\hat{A}}\left(\partial_{[\mu}E_{\nu]}{}^{\hat{B}]} - \frac{1}{4}\delta^{ij}\overline{\psi}_{[\mu i}\gamma^{\hat{B}]}\psi_{\nu]j}\right)$$

(48)

$$+E_{\mu\hat{C}}E^{\nu\hat{A}}E^{\rho\hat{B}}\left(\partial_{[\nu}E_{\rho]}\hat{C} - \frac{1}{4}\delta^{ij}\overline{\psi}_{[\nu i}\gamma^{\hat{C}}\psi_{\rho]j}\right).$$

From this expression, one derives that the supersymmetry transformation of the (dependent) spin connection is given by

从该表达式可推导出 (依赖型) 自旋联络的超对称变换为

$$\delta\Omega_\mu{}^{\hat{A}\hat{B}}(E, \Psi_i) = -\frac{1}{2}\delta^{ij}E^{\nu[\hat{A}}\overline{\eta}_i\gamma^{\hat{B}]}\widehat{\psi}_{\mu\nu j} + \frac{1}{4}\delta^{ij}E_{\mu\hat{C}}E^{\nu\hat{A}}E^{\rho\hat{B}}\overline{\eta}_i\gamma^{\hat{C}}\widehat{\psi}_{\nu\rho j},$$

(49)where $\widehat{\Psi}_{\mu\nu i} \equiv 2D_{[\mu}\Psi_{\nu]i}$. Note that this transformation rule is zero on-shell, i.e., it vanishes upon using the fermionic equations of motion

其中 $\widehat{\Psi}_{\mu\nu i} \equiv 2D_{[\mu}\Psi_{\nu]i}$ 。注意该变换规则在壳上为零，即利用费米子运动方程后它消失

$$\widehat{\psi}_{\mu\nu i} = 0. \tag{50}$$

One may verify that with the transformation rules (46) and (47) the supersymmetry algebra closes on-shell on the fields.

可以验证，借助变换规则 (46) 和 (47)，超对称代数在场上壳闭合。

Besides the relativistic $\mathcal{N} = 2$ supergravity multiplet, we will introduce an additional field M_μ , which can be associated to the central charge generator $\mathcal{Z}$ of the $\mathcal{N} = 2$ super-Poincaré algebra (see Eq. (55)). This gauge field transforms under diffeomorphisms and abelian gauge transformations (with parameter Λ) as follows:

除相对论性 $\mathcal{N} = 2$ 超引力多重态外，我们还将引入一个附加场 M_μ ，它可对应 $\mathcal{N} = 2$ 超庞加莱代数的中心荷生成元 $\mathcal{Z}$ (见式 (55))。该规范场在微分同胚和阿贝尔规范变换 (参数为 Λ) 下的变换如下:

$$\delta M_\mu = \xi^\nu\partial_\nu M_\mu + \partial_\mu\xi^\nu M_\nu + \partial_\mu\Lambda. \tag{51}$$

Its transformation rule under supersymmetry is determined by the Poincaré superal-gebra (see Eq. (55))

它的超对称变换规则由庞加莱超代数确定 (见式 (55))

$$\delta M_\mu = \frac{1}{2}\varepsilon^{ij}\overline{\eta}_i \Psi_{\mu j}. \tag{52}$$

This field is ordinarily not introduced in the supergravity multiplet. In order not to upset the on-shell counting of bosonic and fermionic degrees of freedom, we are thus obliged to set the supercovariant curvature of M_μ to zero, i.e.,

> 该场通常不会被引入超引力多重态中。为了不破坏玻色子和费米子自由度的壳上计数，我们必须将 M_μ 的超协变曲率设为零，即

$$\widehat{F}_{\mu\nu}(M) \equiv 2\partial_{[\mu}M_{\nu]} - \frac{1}{2}\varepsilon^{ij}\overline{\psi}_{[\mu i}\psi_{\nu]j} = 0, \tag{53}$$

so that this field corresponds to a pure gauge degree of freedom. Note that this constraint also implies that the commutator of two supersymmetry transformations acting on M_μ closes to a general coordinate transformation and a central charge transformation. Moreover, this constraint will be important to obtain finite expressions for the NR spin connections by taking the limit of the relativistic connection. Starting from expression (53), the full set of relativistic equations of motion is obtained by the following chain of supersymmetry transformations:

> 因此该场对应纯规范自由度。注意该约束还意味着，作用在 M_μ 上的两个超对称变换的对易子闭合为一个广义坐标变换和一个中心荷变换。此外，该约束对于通过对相对论联络取极限得到非相对论自旋联络的有限表达式至关重要。从表达式 (53) 出发，全套相对论运动方程可通过如下超对称变换链得到:

$$\widehat{F}_{\mu\nu}(M) = 0 \;\rightarrow\; \widehat{\Psi}_{\mu\nu i} = 0 \;\rightarrow\; \widehat{R}_{\mu\nu}{}^{\hat{A}\hat{B}}(\Omega) = 0. \tag{54}$$

This concludes the summary of our relativistic starting point.

> 至此我们完成了对相对论出发点的总结。

The non-Lorentzian limit procedure employs a redefinition of the fields that mimics the Inönii-Wigner contraction of the $\mathcal{N} = 2$ Poincaré superalgebra to the $\mathcal{N} = 2$ Bargmann superalgebra. To motivate this redefinition, we will first consider this algebra contraction. The starting relativistic superalgebra is given by the following $\mathcal{N} = 2$ Poincaré superalgebra with translation $P_{\hat{A}}$, Lorentz transformations $M_{\hat{A}\hat{B}}$, central extension $\mathcal{Z}$ and supercharges Q^i $(i = 1, 2)$:

> 非洛伦兹极限过程对场做重新定义，这一重新定义模仿了从 $\mathcal{N} = 2$ 庞加莱超代数到 $\mathcal{N} = 2$ 巴格曼超代数的伊诺尼-维格纳收缩。为了说明该重新定义的动机，我们首先讨论这种代数收缩。我们出发点的相对论超代数是如下带有平移 $P_{\hat{A}}$ 、洛伦兹变换 $M_{\hat{A}\hat{B}}$ 、中心扩张 $\mathcal{Z}$ 和超荷 Q^i $(i = 1, 2)$ 的 $\mathcal{N} = 2$ 庞加莱超代数:

$$[M_{\hat{A}\hat{B}}, P_{\hat{C}}] = -2\eta_{\hat{C}[\hat{A}}P_{\hat{B}]},\ [M_{\hat{A}\hat{B}}, M_{\hat{C}\hat{D}}] = 4\eta_{[\hat{A}[\hat{C}]}M_{\hat{D}]\hat{B}]},$$

$$[M_{\hat{A}\hat{B}}, Q^i] = -\frac{1}{2}\gamma_{\hat{A}\hat{B}}Q^i,\ \{Q^i, Q^j\} = -\gamma^{\hat{A}}C^{-1}P_{\hat{A}}\delta^{ij} + C^{-1}\mathcal{Z}\varepsilon^{ij}.$$

(55)

Here, the supercharges Q^i are two-component Majorana spinors. For the gamma-matrices, we choose a real basis, i.e., $\gamma^{\hat{A}} = (i\sigma_2, \sigma_1, \sigma_3)$ and the charge conjugation matrix is taken to be $C = i\gamma^0$.

此处，超荷 Q^i 是两分量马约拉纳旋量。对于伽马矩阵，我们选取实基，即 $\gamma^{\hat{A}} = (i\sigma_2, \sigma_1, \sigma_3)$ ，且电荷共轭矩阵取为 $C = i\gamma^0$ 。

In order to define the Inönü-Wigner contraction, we first define the projections

为了定义伊诺尼-维格纳收缩，我们首先定义投影

$$Q_\pm = \frac{1}{\sqrt{2}}\left(Q^1 \pm \gamma_0 Q^2\right), \tag{56}$$

and split the three-dimensional flat indices $\hat{A}, \hat{B}$ into time-like and space-like indices $\{0, a\}$. We set $M_{ab} = J_{ab}$ for the purely spatial rotations. Next, we perform the following invertible redefinition of the generators:

并将三维平指标 $\hat{A}, \hat{B}$ 拆分为类时指标和类空指标 $\{0, a\}$ 。我们对纯空间旋转设定 $M_{ab} = J_{ab}$ 。接下来，我们对生成元进行如下可逆重新定义:

$$Q_- \to \sqrt{\omega} Q_-,\ Q_+ \to \frac{1}{\sqrt{\omega}} Q_+,\ M_{a0} \to \omega G_a,$$

$$\mathcal{Z} \to -\omega M + \frac{1}{2\omega} H,\ P_0 \to \omega M + \frac{1}{2\omega} H,$$

(57)

where ω is a finite dimensionless contraction parameter, and we leave the generators P_a and J_{ab} untouched. Using these redefinitions, the $\mathcal{N} = 2$ supersymmetric extension of the Bargmann algebra is then obtained in the limit $\omega \to \infty$. In particular, we find the following non-vanishing commutation relations:

其中 ω 是有限无量纲收缩参数，我们不改变生成元 P_a 和 J_{ab} 。通过这些重新定义，在极限 $\omega \to \infty$ 下即可得到 Bargmann 代数的 $\mathcal{N} = 2$ 超对称扩展。我们特别得到以下非对易关系:

$$[J_{ab}, P_c] = -2\delta_{c[a} P_{b]},\ [J_{ab}, G_c] = -2\delta_{c[a} G_{b]},$$

$$[G_a, H] = -P_a,\ [G_a, P_b] = -\delta_{ab} M,$$

$$[J_{ab}, Q_\pm] = -\frac{1}{2}\gamma_{ab} Q_\pm,\ [G_a, Q_+] = -\frac{1}{2}\gamma_{a0} Q_-, \tag{58}$$

$$\{Q_+, Q_+\} = -\gamma^0 C^{-1} H,\ \{Q_+, Q_-\} = -\gamma^a C^{-1} P_a,$$

$$\{Q_-, Q_-\} = -2\gamma^0 C^{-1} M$$

The bosonic part of the algebra corresponds to the Bargmann algebra (10). Note that, since we are working in three dimensions, the spatial rotations are Abelian.

该代数的玻色子部分对应 Bargmann 代数 (10)。注意由于我们在三维框架下讨论，空间旋转是阿贝尔的。

We now extend the above algebra contraction to the fields of the on-shell $\mathcal{N}=2$ supergravity multiplet. For the bosonic fields, we employ the following redefinitions:

我们现在将上述代数收缩扩展到壳 $\mathcal{N}=2$ 超引力多重态的场。对于玻色场，我们采用如下重新定义:

$$E_\mu{}^0 = \omega\tau_\mu + \frac{1}{2\omega}m_\mu,\; E_\mu{}^a = e_\mu{}^a, \tag{59}$$

$$M_\mu = \omega\tau_\mu - \frac{1}{2\omega}m_\mu. \tag{60}$$

In the limit $\omega \to \infty$, the fields $\tau_\mu, e_\mu{}^a$ and m_μ become the clock form, spatial vielbein and central charge gauge field of NC geometry, respectively. To derive their bosonic NR transformation rules (11), we first express the new fields in terms of the old ones, i.e.,:

在极限 $\omega \to \infty$ 下，场 $\tau_\mu, e_\mu{}^a$ 和 m_μ 分别成为 NC 几何的时钟形式、空间标架和中心荷规范场。为推导它们的玻色非相对论变换规则 (11)，我们首先用旧场表示新场，即:

$$\tau_\mu = \frac{1}{2\omega}\left(E_\mu{}^0 + M_\mu\right),\; m_\mu = \omega\left(E_\mu{}^0 - M_\mu\right). \tag{61}$$

By also redefining the symmetry parameters $\Lambda^{ab}, \Lambda^{a0}$ and Λ as

我们还对对称参数 $\Lambda^{ab}, \Lambda^{a0}$ 和 Λ 做重新定义如下

$$\lambda^{ab} = \Lambda^{ab},\; \lambda^a = \omega\Lambda^a{}_0,\; \sigma = -\omega\Lambda, \tag{62}$$

it is straightforward to obtain the bosonic transformation rules (11). All fields transform under diffeomorphisms in the usual way.

即可直接推导出玻色变换规则 (11)。所有场都按常规方式做微分同胚变换。

The redefinitions of the gravitini follow from the way we contract the fermionic generators of the $3D\mathcal{N}=2$ Poincaré superalgebra to get the Bargmann superalgebra, i.e., we first define projected spinors

引力微子的重新定义由我们收缩 $3D\mathcal{N}=2$ 庞加莱超代数的费米生成元得到 Bargmann 超代数的方式给出，即我们首先定义投影旋量

$$\Psi_\pm = \frac{1}{\sqrt{2}}\left(\Psi_1 \pm \gamma_0\Psi_2\right), \tag{63}$$

and we similarly define parameters $\eta_\pm$ from the $\eta_{1,2}$. We then introduce the scalings:

我们类似地从 $\eta_{1,2}$ 得到参数 $\eta_\pm$ ，随后引入标度变换:

$$\Psi_{\mu+} = \sqrt{\omega}\psi_{\mu+},\ \eta_+ = \sqrt{\omega}\varepsilon_+,$$

$$\Psi_{\mu-} = \frac{1}{\sqrt{\omega}}\psi_{\mu-},\ \eta_- = \frac{1}{\sqrt{\omega}}\varepsilon_-. \tag{64}$$

The following NR supersymmetry transformation rules then follow

由此得到如下非相对论超对称变换规则

$$\delta\tau_\mu = \frac{1}{2}\bar{\varepsilon}_+\gamma^0\psi_{\mu+}$$

$$\delta e_\mu{}^a = \frac{1}{2}\bar{\varepsilon}_+\gamma^a\psi_{\mu-} + \frac{1}{2}\bar{\varepsilon}_-\gamma^a\psi_{\mu+}, \tag{65}$$

$$\delta m_\mu = \bar{\varepsilon}_-\gamma^0\psi_{\mu-},$$

as well as

以及

$$\delta\psi_{\mu+} = \partial_\mu\varepsilon_+ - \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\varepsilon_+, \tag{66}$$

$$\delta\psi_{\mu-} = \partial_\mu\varepsilon_- - \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\varepsilon_- + \frac{1}{2}\omega_\mu{}^a\gamma_{a0}\varepsilon_+.$$

The transformation rules of the spinors with respect to the NR bosonic symmetries are found to be

旋量关于非相对论玻色对称性的变换规则为

$$\delta\psi_{\mu+} = \frac{1}{4}\lambda^{ab}\gamma_{ab}\psi_{\mu+}, \tag{67}$$

$$\delta\psi_{\mu-} = \frac{1}{4}\lambda^{ab}\gamma_{ab}\psi_{\mu-} - \frac{1}{2}\lambda^a\gamma_{a0}\psi_{\mu+}.$$

It is understood that the spin connections $\omega_\mu{}^a, \omega_\mu{}^{ab}$ in (66) are dependent, i.e., $\omega_\mu{}^a = \omega_\mu{}^a(e,\tau,m,\psi_\pm)$ and $\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e,\tau,m,\psi_\pm)$. Their explicit expressions are given by

这里 (66) 中的自旋联络 $\omega_\mu{}^a, \omega_\mu{}^{ab}$ 是依赖量，即满足 $\omega_\mu{}^a = \omega_\mu{}^a(e,\tau,m,\psi_\pm)$ 和 $\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e,\tau,m,\psi_\pm)$。它们的显式表达式由下式给出

$$\omega_\mu{}^{ab}(e,\tau,m,\psi_\pm) = -2e^{[a|\upsilon|]}\left(\partial_{[\mu}e_{\upsilon]}{}^{b]} - \frac{1}{2}\overline{\psi}_{[\mu+}\gamma^{b]}\psi_{\upsilon]-}\right)$$

$$+e_\mu{}^c e^{a\upsilon}e^{b\rho}\left(\partial_{[\upsilon}e_{\rho]}{}^c - \frac{1}{2}\overline{\psi}_{[\upsilon+}\gamma^c\psi_{\rho]-}\right)$$

$$-\tau_\mu e^{a\upsilon}e^{b\rho}\left(\partial_{[\upsilon}m_{\rho]} - \frac{1}{2}\overline{\psi}_{[\upsilon-}\gamma^0\psi_{\rho]-}\right), \tag{68}$$

$$\omega_\mu{}^a(e,\tau,m,\psi_\pm) = \tau^\upsilon\left(\partial_{[\mu}e_{\upsilon]}{}^a - \frac{1}{2}\overline{\psi}_{[\mu+}\gamma^a\psi_{\upsilon]-}\right)$$

$$+e_{\mu b}e^{a\upsilon}\tau^\rho\left(\partial_{[\upsilon}e_{\rho]}{}^b - \frac{1}{2}\overline{\psi}_{[\upsilon+}\gamma^b\psi_{\rho]-}\right)$$

$$+e^{a\upsilon}\left(\partial_{[\mu}m_{\upsilon]} - \frac{1}{2}\overline{\psi}_{[\mu-}\gamma^0\psi_{\upsilon]-}\right)$$

$$-\tau_\mu e^{a\upsilon}\tau^\rho\left(\partial_{[\upsilon}m_{\rho]} - \frac{1}{2}\overline{\psi}_{[\upsilon-}\gamma^0\psi_{\rho]-}\right). \tag{69}$$

The expressions for these NR spin connections can be obtained from the relativistic expression given in Eq. (48), by also using Eq. (53). In particular, in order to obtain these expressions, we have used Eq. (53), for finite ω , to replace terms that diverge in the $\omega \to \infty$ limit, by terms with the expected ω -order. Note that the expressions (68), (69) are of the form (23) with

这些非相对论自旋联络的表达式可以从式 (48) 给出的相对论表达式结合式 (53) 得到。特别地，为得到这些表达式，我们对有限 ω 利用了式 (53)，将在 $\omega \to \infty$ 极限下发散的项替换为符合预期 ω 阶的项。注意式 (68)、(69) 符合式 (23) 的形式，满足

$$T_{\mu\upsilon}{}^a = \overline{\psi}_{[\mu+}\gamma^a\psi_{\upsilon]-} \quad\text{and}\quad T^{(m)}_{\mu\upsilon} = \overline{\psi}_{[\mu-}\gamma^0\psi_{\upsilon]-}. \tag{70}$$

According to the discussion that led to Eq. (23), the dependent spin connections (68), (69) thus identically fulfil the following conventional constraints:

根据推导式 (23) 的讨论，依赖自旋联络 (68)、(69) 因此恒满足如下常规约束:

$$\widehat{R}_{\mu\upsilon}{}^a(P) \equiv R_{\mu\upsilon}{}^a(P) - \overline{\psi}_{[\mu+}\gamma^a\psi_{\upsilon]-} = 0,$$

(71)

$$\widehat{R}_{\mu\upsilon}(M) \equiv R_{\mu\upsilon}(M) - \overline{\psi}_{[\mu-}\gamma^0\psi_{\upsilon]-} = 0,$$

where $R_{\mu\upsilon}{}^a(P)$ and $R_{\mu\upsilon}(M)$ are defined in (22) and where $\widehat{R}_{\mu\upsilon}{}^a(P)$ and $\widehat{R}_{\mu\upsilon}(M)$ are the supercovariant curvatures of $e_\mu{}^a$ and m_μ , respectively. The other superco-variant curvatures that will be used in what follows are given by

其中 $R_{\mu\upsilon}{}^a(P)$ 和 $R_{\mu\upsilon}(M)$ 由 (22) 定义，$\widehat{R}_{\mu\upsilon}{}^a(P)$ 和 $\widehat{R}_{\mu\upsilon}(M)$ 分别是 $e_\mu{}^a$ 和 m_μ 的超协变曲率。下文将要用到的其他超协变曲率列出如下

$$\widehat{R}_{\mu\upsilon}{}^a(G) \equiv 2\partial_{[\mu}\omega_{\upsilon]}{}^a - 2\omega_{[\mu}{}^{ab}\omega_{\upsilon]b}$$

$$\widehat{R}_{\mu\upsilon}{}^{ab}(J) \equiv 2\partial_{[\mu}\omega_{\upsilon]}{}^{ab},$$

$$\widehat{R}_{\mu\upsilon}(H) \equiv 2\partial_{[\mu}\tau_{\upsilon]} - \frac{1}{2}\overline{\psi}_{[\mu+}\gamma^0\psi_{\upsilon]+},$$

$$\widehat{\psi}_{\mu\upsilon+} \equiv 2\partial_{[\mu}\psi_{\upsilon]+} - \frac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\upsilon]+},$$

$$\widehat{\psi}_{\mu\upsilon-} \equiv 2\partial_{[\mu}\psi_{\upsilon]-} - \frac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\upsilon]-} + \omega_{[\mu}{}^{a}\gamma_{a0}\psi_{\upsilon]+}. \tag{72}$$

The conventional constraints (71) can be used to determine that the supersymme-try transformations of the dependent spin and boost connections (68) and (69) are given by

> 利用常规约束 (71) 可以确定，依赖自旋与 boost 联络 (68) 和 (69) 的超对称变换形式为

$$\delta_Q\omega_\mu{}^{ab}(e,\tau,m,\psi_\pm) = \frac{1}{2}\bar{\varepsilon}_+\gamma^{[b}\widehat{\psi}^{a]}{}_{\mu-} + \frac{1}{4}e_{\mu c}\bar{\varepsilon}_+\gamma^c\widehat{\psi}^{ab}{}_- - \frac{1}{2}\tau_\mu\bar{\varepsilon}_-\gamma^0\widehat{\psi}^{ab}{}_-$$

$$+\frac{1}{2}\bar{\varepsilon}_-\gamma^{[b}\widehat{\psi}^{a]}{}_{\mu+} + \frac{1}{4}e_{\mu c}\bar{\varepsilon}_-\gamma^c\widehat{\psi}^{ab}{}_+,$$

(73)

$$\delta_Q\omega_\mu{}^{a}(e,\tau,m,\psi_\pm) = \frac{1}{2}\bar{\varepsilon}_-\gamma^0\widehat{\psi}_\mu{}^a - +\frac{1}{2}\tau_\mu\bar{\varepsilon}_-\gamma^0\widehat{\psi}_0{}^a - +\frac{1}{4}e_{\mu b}\bar{\varepsilon}_+\gamma^b\widehat{\psi}^a{}_{0-}$$

$$+\frac{1}{4}\bar{\varepsilon}_+\gamma^a\widehat{\psi}_{\mu 0-} + \frac{1}{4}e_{\mu b}\bar{\varepsilon}_-\gamma^b\widehat{\psi}^a{}_{0+} + \frac{1}{4}\bar{\varepsilon}_-\gamma^a\widehat{\psi}_{\mu 0+}.$$

(74)

The constraint (53) is not only used to get finite expressions for the NR spin connections. As already alluded to above, its $\omega \to \infty$ limit also leads to the following constraint:

> 约束 (53) 不仅用于得到非相对论自旋联络的有限表达式，正如前文已经暗示的，它的 $\omega \to \infty$ 极限还给出了如下约束:

$$\widehat{R}_{\mu\nu}(H) = 0. \tag{75}$$

This constraint is a geometric one (i.e., not a conventional one) and leads to further conditions upon variation under supersymmetry. One finds that under supersymmetry transformations, with parameters ε_+ and ε_- , the following set of constraints is generated:

> 该约束是几何约束 (即非常规约束)，在超对称变分会产生额外条件。可以发现，在参数为 ε_+ 和 ε_- 的超对称变换下，会生成下列一组约束:

$$\widehat{\psi}_{ab-} = 0 \tag{76}$$

$$\widehat{R}_{\mu\nu}(H) = 0 \;\rightarrow\; \widehat{\psi}_{\mu\nu+} = 0 \;\rightarrow\; \widehat{R}_{\mu\nu}{}^{ab}(J) = 0 \tag{77}$$

$$\gamma^a\widehat{\psi}_{a0-} = 0 \;\rightarrow\; \widehat{R}_{0a}{}^{a}(G) = 0.$$

(78)

Note that the variation of the $\widehat{\psi}_{\mu\nu+} = 0$ constraint leads to three different constraints. Of these three constraints, only the variation of the constraint $\gamma^a \widehat{\psi}_{a0-} = 0$ leads to one more constraint. Using the last constraint given in Eq. (77), the NR Bianchi identities reduce to

请注意，$\widehat{\psi}_{\mu\nu+} = 0$ 约束的变分会得到三个不同的约束。在这三个约束中，只有约束 $\gamma^a \widehat{\psi}_{a0-} = 0$ 的变分会额外生成一个约束。利用式 (77) 给出的最后一个约束，非相对论比安基恒等式可化简为

$$\widehat{R}_{ab}{}^{c}(G) = 0, \ \widehat{R}_{0[a}{}^{b]}(G) = 0. \tag{79}$$

These identities are needed to show that the variation of the constraint given in Eq. (76) does not lead to further constraints and that the variation of the first constraint in (78) leads to the singlet constraint $R_{0a}{}^{a}(G) = 0$ on the boost curvature. As we will see in the next subsection, this singlet constraint corresponds to a covariant (i.e., valid in arbitrary reference frames) version of the Poisson equation for the Newton potential.

这些恒等式可以用来证明，式 (76) 给出的约束的变分不会产生额外约束，且 (78) 中第一个约束的变分会得到 boost 曲率上的单态约束 $R_{0a}{}^{a}(G) = 0$ 。我们将在下一小节看到，该单态约束对应牛顿势泊松方程的协变 (即适用于任意参考系) 形式。

At this point, we have obtained the supersymmetry rules of all gauge fields, both the dependent and the independent ones. We find that with these supersymmetry transformations the supersymmetry algebra closes on-shell. To be precise, the commutator of two supersymmetry transformations closes and is given by the following soft algebra:

至此，我们已经得到了所有规范场 (包括依赖规范场和独立规范场) 的超对称规则。我们发现，结合这些超对称变换，超对称代数在壳闭合。准确来说，两个超对称变换对易子闭合，由如下软代数给出:

$$\left[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)\right] = \delta_{\text{g.c.t.}}(\xi^\mu) + \delta_{J_{ab}}(\lambda^a{}_b) + \delta_{G_a}(\lambda^a) + \delta_{Q_+}(\varepsilon_+) +$$

$$+\delta_{Q_-}(\varepsilon_-) + \delta_M(\sigma) \tag{80}$$

provided the two fermionic constraints on the gravitino curvature in Eqs. (76) and (78) hold. Here, g.c.t. denotes a general coordinate transformation, and the field-dependent parameters are given by

前提是式 (76) 和 (78) 中引力微子曲率的两个费米子约束成立。此处 g.c.t. 表示广义坐标变换，依赖场的参数由下式给出

$$\xi^\mu = \frac{1}{2}\left(\bar{\varepsilon}_{2+}\gamma^0\varepsilon_{1+}\right)\tau^\mu + \frac{1}{2}\left(\bar{\varepsilon}_{2+}\gamma^a\varepsilon_{1-} + \bar{\varepsilon}_{2-}\gamma^a\varepsilon_{1+}\right)e_a{}^\mu,$$

$$\lambda^a{}_b = -\xi^\mu\omega_\mu{}^a{}_b$$

$$\lambda^a = -\xi^\mu\omega_\mu{}^a$$

$$\varepsilon_{\pm} = -\xi^{\mu}\psi_{\mu\pm}$$

$$\sigma = -\xi^{\mu}m_{\mu} + \left(\bar{\varepsilon}_{2-}\gamma^{0}\varepsilon_{1-}\right). \tag{81}$$

This concludes our discussion of how by taking a limit one can derive the three-dimensional on-shell $\mathcal{N} = 2$NC supergravity theory first constructed in [10] by gauging a $\mathcal{N} = 2$ supersymmetric extension of the Bargmann algebra, i.e., we obtained all constraints, equations of motion and transformation rules. To finish the consistency check of our procedure, we should check whether the supersymmetry variation of the bosonic equation of motion given in (78), describing the Poisson equation of the Newton potential in arbitrary frames, does not lead to new constraints and/or equations of motion. Instead of doing this, we shall show in the next subsection that all constraints can be solved after gauge fixing, leading to a consistent system with a closed algebra.

至此我们完成了推导: 通过取极限，我们得到了三维在壳 $\mathcal{N} = 2$NC 超引力理论，该理论最早由文献 [10] 通过给 Bargmann 代数的 $\mathcal{N} = 2$ 超对称扩展做规范构造得到，也就是我们已经得到了所有约束、运动方程和变换规则。为了完成对我们推导流程的一致性检验，我们应当验证: 描述任意参考系下牛顿势泊松方程的、式 (78) 给出的玻色子运动方程，其超对称变分不会产生新的约束和/或运动方程。我们不直接做这项检验，而是在下一小节说明，所有约束都可以在规范固定后求解，最终得到一个代数封闭的自洽体系。

Gauge Fixing

规范固定

For clarity, we will first explain how the gauge fixing of local diffeomorphisms, spatial rotations, Galilean boosts and the central charge transformation works in the purely bosonic case with all fermions set to zero. After that, we will extend the discussion and include local supersymmetry and the fermions in the gauge fixing procedure.

为清晰起见，我们首先说明在所有费米子都设为零的纯玻色子情形中，局部微分同胚、空间旋转、伽利略 boost 和中心荷变换的规范固定是如何运作的。在此之后，我们会拓展讨论，将局部超对称和费米子纳入规范固定过程中。

The Bosonic Case

玻色子情形

We start by first solving the two bosonic constraints in (77), with all fermions set to zero, by imposing the gauge fixing conditions (We use a notation where $\varnothing$ indicates a curved $\mu = 0$ index.)

我们首先从求解式 (77) 中的两个玻色约束开始，此时所有费米子都已置零，通过施加规范固定条件 (我们采用的记号中，$\varnothing$ 表示弯曲的 $\mu = 0$ 指标。)

$$\tau_\mu(x^\nu) = \delta_\mu{}^\varnothing,\ \omega_\mu{}^{ab}(x^\nu) = 0. \tag{82}$$

This fixes the local time translations and spatial rotations to constant ones:

这将局域时间平移和空间转动固定为常数变换:

$$\xi^\varnothing(x^\nu) = \xi^\varnothing,\ \lambda^{ab}(x^\nu) = \lambda^{ab}. \tag{83}$$

No compensating transformations are induced by these gauge fixings.

这些规范固定不会诱导出补偿变换。

Next, we gauge fix the spatial dependence of the spatial translations by imposing the gauge fixing condition

接下来，我们通过施加规范固定条件来规范固定空间平移的空间依赖关系

$$e_i^a(x^\nu) = \delta_i^a. \tag{84}$$

Requiring $\delta e_i{}^a = 0$ leads to the condition (with $t = x^0$)

要求 $\delta e_i{}^a = 0$ 可得到条件 (其中 $t = x^0$)

$$\xi^a(x^\nu) = \xi^a(t) - \lambda_{ai}x^i. \tag{85}$$

Note that after imposing the gauge fixing (84), the spatial part of space-time corresponds to a flat, Euclidean space. There is thus no longer any need to distinguish the i and a indices and upper and down indices, and we will not do so in the following.

注意，施加规范固定条件 (84) 后，时空的空间部分对应一个平坦欧几里得空间，因此不再需要区分 i 指标、a 指标以及上下指标，下文中我们将不再区分。

At this stage, the independent time-like and spatial Dreibein components and their projective inverses are given by

在此阶段，独立的类时和空间三元基分量及其投影逆由下式给出

$$\tau_\mu(x^\nu) = \delta_\mu{}^\varnothing,\ e_\mu{}^a(x^\nu) = (-\tau^a(x^\nu), \delta_i{}^a),$$

$$\tau^\mu(x^\nu) = (1, \tau^a(x^\nu)),\ e_a{}^\mu(x^\nu) = \left(0, \delta^i{}_a\right), \tag{86}$$

where the $\tau^a(x^\nu)$ are the only non-constant Dreibein components left. The only other independent gauge field left is the central charge gauge field $m_\mu(x^\nu)$. Taking into account the compensating gauge transformation given in (85), we find that the remaining independent fields $\tau^a(x^\nu)$, $m_\varnothing(x^\nu)$ and $m_i(x^\nu)$ transform as follows under the leftover transformations:

其中 $\tau^a(x^\upsilon)$ 是仅剩的非恒定三元基分量。剩下唯一的其他独立规范场是中心荷规范场 $m_\mu(x^\upsilon)$。考虑到式 (85) 给出的补偿规范变换，我们得到剩余独立场 $\tau^a(x^\upsilon)$, $m_\varnothing(x^\upsilon)$ 和 $m_i(x^\upsilon)$ 在剩余变换下的变换规则如下:

$$\delta\tau^a(x^\upsilon) = \lambda^a{}_b\tau^b(x^\upsilon) - \lambda^c{}_d x^d\partial_c\tau^a(x^\upsilon) + \xi^\varnothing\partial_\varnothing\tau^a(x^\upsilon) + \xi^j(t)\,\partial_j\tau^a(x^\upsilon) -$$

$$-\dot{\xi}^a(t) - \lambda^a(x^\nu) \tag{87}$$

$$\delta m_i(x^\upsilon) = \xi^\varnothing\partial_\varnothing m_i(x^\upsilon) + \xi^j(t)\,\partial_j m_i(x^\upsilon) + \lambda_i{}^j m_j(x^\upsilon) - \lambda^j{}_k x^k\partial_j m_i(x^\upsilon) +$$

$$+\lambda_i(x^\upsilon) + \partial_i\sigma(x^\upsilon) \tag{88}$$

$$\delta m_\varnothing(x^\upsilon) = \xi^\varnothing\partial_\varnothing m_\varnothing(x^\upsilon) + \dot{\xi}^i(t)\,m_i(x^\upsilon) + \xi^i(t)\,\partial_i m_\varnothing(x^\upsilon) - \lambda^i{}_j x^j\partial_i m_\varnothing(x^\upsilon) -$$

$$-\lambda^a(x^\upsilon)\,\tau_a(x^\upsilon) + \dot{\sigma}(x^\upsilon), \tag{89}$$

where denotes a derivative with respect to t .

其中表示对 t 的导数。

The three fields $\tau^a(x^\upsilon)$, $m_i(x^\upsilon)$ and $m_\varnothing(x^\upsilon)$ are not independent. Since the gauge field $\omega_\mu{}^{ab}(x^\upsilon)$, which we gauge fixed to zero (see Eq. (82)), is dependent, we need to investigate its consequences. Using the other gauge fixing conditions as well, we find that the condition $\omega_\varnothing{}^{ab}(x^\upsilon) = 0$ leads to the following restriction:

三个场 $\tau^a(x^\upsilon)$, $m_i(x^\upsilon)$ 和 $m_\varnothing(x^\upsilon)$ 并非独立。由于我们已经规范固定为零的规范场 $\omega_\mu{}^{ab}(x^\upsilon)$ (见式 (82)) 是依赖场，我们需要研究它的影响。再结合其他规范固定条件，我们得到条件 $\omega_\varnothing{}^{ab}(x^\upsilon) = 0$ 给出如下限制:

$$\partial_{[i}\tau_{j]}(x^\upsilon) + \partial_{[i}m_{j]}(x^\upsilon) = 0. \tag{90}$$

This implies that, locally, one can write

这意味着，在局域上可以写为

$$\tau_i(x^\upsilon) + m_i(x^\upsilon) = \partial_i m(x^\upsilon). \tag{91}$$

Without loss of generality, we can thus eliminate $m_i(x^\upsilon)$ for $\tau_i(x^\upsilon)$ and $m(x^\upsilon)$, which is what we will do in the following. The transformation rule for $m(x^\upsilon)$ can be found from $\delta\tau_i(x^\upsilon)$ and $\delta m_i(x^\upsilon)$:

因此，不失一般性，我们可以用 $\tau_i(x^\upsilon)$ 和 $m(x^\upsilon)$ 消去 $m_i(x^\upsilon)$，下文中我们将采用这一处理。$m(x^\upsilon)$ 的变换规则可由 $\delta\tau_i(x^\upsilon)$ 和 $\delta m_i(x^\upsilon)$ 得到:

$$\delta m(x^{\upsilon}) = \xi^{\varnothing}\partial_{\varnothing}m(x^{\upsilon}) - \dot{\xi}^{k}(t)x^{k} + \xi^{j}(t)\partial_{j}m(x^{\upsilon}) - \lambda^{j}{}_{k}x^{k}\partial_{j}m(x^{\upsilon}) + \sigma(x^{\upsilon}) + Y(t),$$

(92)

where $Y(t)$ is an arbitrary time-dependent shift. At this point, we are left with three independent fields $\tau^{i}(x^{\upsilon}), m_{\varnothing}(x^{\upsilon})$ and $m(x^{\upsilon})$ whose transformation laws are given by (87), (89), (92), respectively.

其中 $Y(t)$ 是任意含时平移。至此，我们仅剩三个独立场 $\tau^{i}(x^{\upsilon}), m_{\varnothing}(x^{\upsilon})$ 和 $m(x^{\upsilon})$，它们的变换规律分别由 (87), (89), (92) 给出。

From the transformation rule (92), we see that the central charge transformation, with local parameter $\sigma(x^{\upsilon})$, acts as a Stückelberg shift on the field $m(x^{\upsilon})$. We can thus partially fix the central charge transformations by imposing

从变换规则 (92) 可以看出，带有局域参数 $\sigma(x^{\upsilon})$ 的中心荷变换对场 $m(x^{\upsilon})$ 起到施蒂克尔贝格平移的作用。因此我们可以通过施加下式部分固定中心荷变换:

$$m(x^{\upsilon}) = 0\,. \tag{93}$$

This fixes the central charge transformations according to

这按照下式固定了中心荷变换:

$$\sigma(x^{\mu}) = \sigma(t) + \dot{\xi}^{a}(t)x_{a}, \tag{94}$$

where it is understood that we also fix $Y(t) = -\sigma(t)$ in (92). After this gauge fixing, the transformation rules of the two independent fields $\tau^{i}(x^{\upsilon})$ and $m_{\varnothing}(x^{\upsilon})$ are given by:

其中我们已知还需要固定 (92) 式中的 $Y(t) = -\sigma(t)$。完成该规范固定后，两个独立场 $\tau^{i}(x^{\upsilon})$ 和 $m_{\varnothing}(x^{\upsilon})$ 的变换规则如下:

$$\delta\tau^{i}(x^{\upsilon}) = \lambda^{i}{}_{j}\tau^{j}(x^{\upsilon}) - \lambda^{j}{}_{k}x^{k}\partial_{j}\tau^{i}(x^{\upsilon}) + \xi^{\varnothing}\partial_{\varnothing}\tau^{i}(x^{\upsilon}) + \xi^{j}(t)\partial_{j}\tau^{i}(x^{\upsilon}) - \dot{\xi}^{i}(t)$$

$$-\lambda^{i}(x^{\nu})$$

$$\delta m_{\varnothing}(x^{\upsilon}) = \xi^{\varnothing}\partial_{\varnothing}m_{\varnothing}(x^{\upsilon}) - \dot{\xi}^{i}(t)\tau_{i}(x^{\upsilon}) + \xi^{i}(t)\partial_{i}m_{\varnothing}(x^{\upsilon}) + \ddot{\xi}^{k}(t)x^{k}$$

$$-\lambda^{i}{}_{j}x^{j}\partial_{i}m_{\varnothing}(x^{\upsilon}) - \lambda^{i}(x^{\upsilon})\tau_{i}(x^{\upsilon}) + \dot{\sigma}(t)\,. \tag{95}$$

Next, we note that the local boost transformations, with local parameters $\lambda^{i}(x^{\nu})$, also end up as a Stueckelberg symmetry. This Stueckelberg symmetry can be fixed by imposing the final gauge condition

接下来我们注意到，带局部参数 $\lambda^{i}(x^{\nu})$ 的局部 Boost 变换最终也成为施特克尔贝格对称性。该对称性可通过施加最终规范条件来固定

$$\tau^a(x^\upsilon) = 0. \tag{96}$$

This leads to the following compensating transformations:

由此得到以下补偿变换:

$$\lambda^i(x^\upsilon) = -\dot{\xi}^i(t). \tag{97}$$

The only independent field left now is

现在仅存的独立场为

$$m_\varnothing(x^\upsilon) \equiv \Phi(x^\upsilon), \tag{98}$$

which we will soon identify as the Newton potential. Using the gauge condition (96) and taking into account the compensating transformations (97), we find that the transformation rule of this field is given by

我们很快就会将其认定为牛顿势。利用规范条件 (96) 并考虑补偿变换 (97)，我们得到该场的变换规则为

$$\delta\Phi(x^\upsilon) = \xi^\varnothing\partial_\varnothing\Phi(x^\upsilon) + \xi^i(t)\,\partial_i\Phi(x^\upsilon) + \ddot{\xi}^k(t)\,x^k - \lambda^i{}_j x^j\partial_i\Phi(x^\upsilon) + \dot{\sigma}(t). \tag{99}$$

The fact that we identify the field $m_\varnothing(x^\upsilon)$ with the Newton potential $\Phi(x^\upsilon)$ is justified by noting that it satisfies the Poisson equation. To see this, first note that in terms of $\Phi(x^\upsilon)$ the expression for the only non-zero-dependent boost spin connection components (see Eq. (69)) is given by

我们将场 $m_\varnothing(x^\upsilon)$ 认定为牛顿势 $\Phi(x^\upsilon)$ 的依据是它满足泊松方程。要说明这一点，首先注意到，用 $\Phi(x^\upsilon)$ 表示的唯一非零依赖 Boost 自旋联络分量 (见式 (69)) 为

$$\omega_\varnothing^a(x^\upsilon) = -\partial^a\Phi(x^\upsilon). \tag{100}$$

If we now plug this expression for the boost spin connection components into the bosonic equation of motion given in Eq. (78), we find the expected Poisson equation for the Newton potential:

如果我们将这个 Boost 自旋联络分量的表达式代入式 (78) 给出的玻色子运动方程，就能得到牛顿势满足的预期泊松方程:

$$\Delta\Phi = \partial_a\partial_a\Phi = 0. \tag{101}$$

This equation is invariant under the so-called acceleration extended Galilei symmetries (99).

该方程在所谓加速度扩展伽利略对称性 (99) 下保持不变。

The transformations (99) close an algebra on $\Phi(x^\upsilon)$. One finds the following nonzero commutators:

变换 (99) 在 $\Phi\left(x^{\upsilon}\right)$ 上闭合形成一个代数。可得到以下非零对易子:

$$\left[\delta_{\xi^{\varnothing}},\delta_{\xi^{i}(t)}\right]\Phi\left(x^{\upsilon}\right)=\delta_{\xi^{i}(t)}\left(-\xi^{\varnothing}\dot{\xi}^{i}\left(t\right)\right)\Phi\left(x^{\upsilon}\right),$$

$$\left[\delta_{\xi^{\varnothing}},\delta_{\sigma(t)}\right]\Phi\left(x^{\upsilon}\right)=\delta_{\sigma(t)}\left(-\xi^{\varnothing}\dot{\sigma}\left(t\right)\right)\Phi\left(x^{\upsilon}\right),$$

$$\left[\delta_{\xi_{1}^{i}(t)},\delta_{\xi_{2}^{i}(t)}\right]\Phi\left(x^{\upsilon}\right)=\delta_{\sigma(t)}\left(\dot{\xi}_{1}^{j}\left(t\right)\xi_{2}^{j}\left(t\right)-\dot{\xi}_{2}^{j}\left(t\right)\xi_{1}^{j}\left(t\right)\right)\Phi\left(x^{\upsilon}\right),$$

$$\left[\delta_{\xi^{i}(t)},\delta_{\lambda^{jk}}\right]\Phi\left(x^{\upsilon}\right)=\delta_{\xi^{i}(t)}\left(\lambda^{i}{}_{j}\xi^{j}\left(t\right)\right)\Phi\left(x^{\upsilon}\right),\tag{102}$$

where we have indicated the parameters of the transformations on the right-hand side in the brackets.

其中我们已经将变换参数标在右侧括号内。

This finishes our review of the bosonic case. For the convenience of the reader, we have summarized all gauge conditions and resulting compensating transformations in Table 2.

至此我们完成了对玻色子情形的讨论。为方便读者，我们将所有规范条件和得到的补偿变换汇总在表 2 中。

Table 2 This table indicates the gauge fixing conditions and corresponding compensating transformations that allow one to obtain Newtonian gravity, in frames that are rectilinearly accelerating with respect to inertial reference frames, from NC gravity. We have also included the restrictions that follow from the fact that the spin connection field $\omega_{\mu}{}^{ab}$ is dependent. At the bottom of the table, we have summarized the expressions of the nonzero remaining gauge fields in terms of the Newton potential $\Phi\left(x^{\upsilon}\right)$

表 2 本表格列出了从牛顿-卡坦 (NC) 引力得到牛顿引力所需的规范固定条件与对应补偿变换，该推导适用于相对惯性系做匀加速直线运动的参考系。我们还加入了由自旋联络场 $\omega_{\mu}{}^{ab}$ 存在依赖性得到的约束。在表格底部，我们汇总了剩余非零规范场用牛顿势 $\Phi\left(x^{\upsilon}\right)$ 表示的表达式

Gauge condition/restriction	Compensating transformation
$\tau_{\mu}\left(x^{\nu}\right)=\delta_{\mu}{}^{\varnothing}$	$\xi^{\varnothing}\left(x^{\upsilon}\right)=\xi^{\varnothing}$
$\omega_{\mu}{}^{ab}\left(x^{\nu}\right)=0$	$\lambda^{ab}\left(x^{\upsilon}\right)=\lambda^{ab}$
$e_{i}{}^{a}\left(x^{\upsilon}\right)=\delta_{i}{}^{a}$	$\xi^{a}\left(x^{\upsilon}\right)=\xi^{a}\left(t\right)-\lambda_{ai}x^{i}$
$\tau_{i}\left(x^{\upsilon}\right)+m_{i}\left(x^{\upsilon}\right)=\partial_{i}m\left(x^{\upsilon}\right)$	-
$m\left(x^{\nu}\right)=0$	$\sigma\left(x^{\upsilon}\right)=\sigma\left(t\right)+\dot{\xi}^{a}\left(t\right)x_{a}$
$\tau^{a}\left(x^{\upsilon}\right)=0$	$\lambda^{i}\left(x^{\upsilon}\right)=-\dot{\xi}^{i}\left(t\right)$
$m_{\varnothing}\left(x^{\upsilon}\right)=\Phi\left(x^{\upsilon}\right)$	$\omega_{\varnothing}{}^{a}\left(x^{\upsilon}\right)=-\partial^{a}\Phi\left(x^{\upsilon}\right)$

The Supersymmetric Case

超对称情形

Our aim is now to include the fermions and local supersymmetry transformations in the discussion of the previous subsection and to perform a partial gauge fixing of the bosonic and fermionic symmetries to derive the NC supergravity theory from the Galilean observer point of view. We will define a supersymmetric Galilean observer as one for which only a supersymmetric extension of the acceleration extended Galilei symmetries is retained. We will see below that only half of the supersymmetries are gauge fixed to constant ones, due to the fact that the only translations that are gauge fixed to constants are the time translations.

我们现在的目标是把费米子和局域超对称变换纳入上一小节的讨论中，并对玻色对称性和费米对称性做部分规范固定，从伽利略观测者的视角得到牛顿-卡特超引力理论。我们将超对称伽利略观测者定义为仅保留加速度扩展伽利略对称性的超对称推广的观测者。下文将会看到，只有一半超对称被规范固定为常超对称，这是因为只有时间平移被规范固定为常平移。

First, we solve the constraints (77), but now keeping the fermions, by imposing the gauge fixing conditions

首先，我们通过施加规范固定条件求解约束 (77)，本次求解保留费米子

$$\tau_\mu\left(x^\nu\right)=\delta_\mu{}^{\varnothing},\ \omega_\mu{}^{ab}\left(x^\nu\right)=0,\ \psi_{\mu+}\left(x^\nu\right)=0. \tag{103}$$

This fixes the local time translations, spatial rotations and ε_+ transformations to constant transformations:

这将局域时间平移、空间转动和 ε_+ 变换规范固定为常变换:

$$\xi^{\varnothing}\left(x^\nu\right)=\xi^{\varnothing},\ \lambda^{ab}\left(x^\nu\right)=\lambda^{ab},\ \varepsilon_+\left(x^\nu\right)=\varepsilon_+. \tag{104}$$

No compensating transformations are induced by these gauge fixings.

这些规范固定不会诱导补偿变换。

Next, we partially gauge fix the spatial translations by imposing the gauge choice

接下来，我们通过施加规范选择对空间平移做部分规范固定

$$e_i^a\left(x^\nu\right)=\delta_i^a. \tag{105}$$

This gauge choice implies that we may use from now on the expressions (86) for the time-like and spatial Dreibein components and their projective inverses. We will derive the required compensating transformation below. First, using the above gauge choices and the fact that the purely spatial components $\widehat{R}_{ij}{}^a(G)$ of the curvature of boost transformations and the purely spatial components $\widehat{\psi}_{ij-}$ of the curvature of ε_- transformations are zero, we derive from their expressions (see Eq. (72)) that

该规范选择意味着我们从现在起可以将表达式 (86) 用于类时和空间三维标架分量及其投影逆。我们将在之后推导所需的补偿变换。首先，利用上述规范选择，以及 boost 变换曲率的纯空间分量 $\widehat{R}_{ij}{}^{a}(G)$ 和 ε_- 变换曲率的纯空间分量 $\widehat{\psi}_{ij-}$ 均为零这一性质，我们从它们的表达式 (见式 (72)) 推导出:

$$\partial_{[i}\omega_{j]}{}^{a} = 0,\ \partial_{[i}\psi_{j]-} = 0. \tag{106}$$

In the first equation, we solve locally by writing

在第一个方程中，我们通过写下以下形式进行局部求解:

$$\omega_i^a = \partial_i \omega^a, \tag{107}$$

where ω^a is a dependent field since $\omega_i{}^a$ is dependent. This also explains why we have not added a purely time-dependent piece to the r.h.s. of the above solution.

其中 ω^a 是依赖场，因为 $\omega_i{}^a$ 是依赖场。这也解释了为什么我们没有在上文解的右侧添加纯时间依赖项。

We subsequently partially gauge fix the ε_- transformations by imposing the gauge choice

随后我们通过施加规范选择对 ε_- 变换做部分规范固定

$$\psi_{i-}(x^\nu) = 0. \tag{108}$$

This fixes the ε_- transformations according to

根据下式，该操作固定了 ε_- 变换:

$$\varepsilon_-(x^\nu) = \varepsilon_-(t) - \frac{1}{2}\omega^a \gamma_{a0}\varepsilon_+. \tag{109}$$

Given the gauge choice (108), the spatial translations are now fixed without the need for any fermionic compensating transformation. Indeed, from the total variation of the gauge fixing condition (105), we find:

给定规范选择 (108)，空间平移现在已被固定，不需要任何费米补偿变换。实际上，从规范固定条件 (105) 的全变分，我们得到:

$$\xi^i(x^\nu) = \xi^i(t) - \lambda^i{}_j x^j. \tag{110}$$

At this point, we are left with the remaining fields τ^a, m_i, $m_\varnothing$ and $\psi_{\varnothing-}$. These fields are not independent, since the gauge field $\omega_\mu{}^{ab}$, which we gauge fixed to zero, is dependent (see Eq. (68)). Like in the bosonic case, only the gauge fixing $\omega_\varnothing{}^{ab} = 0$ leads to a restriction:

至此，还剩下剩余场 $\tau^a, m_i, m_\varnothing$ 和 $\psi_{\varnothing-}$ 。这些场不是独立的，因为被我们规范固定为零的规范场 ${\omega_\mu}^{ab}$ 本身是依赖场 (见式 (68))。和玻色情形一样，只有规范固定 ${\omega_\varnothing}^{ab} = 0$ 会给出约束:

$$\partial_{[i}\left(\tau_{j]} + m_{j]}\right)(x^\upsilon) = 0. \tag{111}$$

As in the bosonic case, this implies that we can locally write:

和玻色情形一样，这意味着我们可以局部地写为:

$$\tau_i\left(x^\upsilon\right) + m_i\left(x^\upsilon\right) = \partial_i m\left(x^\upsilon\right). \tag{112}$$

Without loss of generality, we will use this equation to eliminate m_i in terms of the other two fields. The variation of m is determined by writing the variation of $\tau_i + m_i$ as a ∂_i -derivative. This is trivial for most of the terms, except for the ε_+ term. Before addressing this issue below, it is convenient to write down the total variation of $\partial_i m$ instead of m . From Eq. (112), we find

不失一般性，我们将利用该式用另外两个场消去 m_i 。通过将 $\tau_i + m_i$ 的变分写为 ∂_i 导数，我们可以得到 m 的变分。除了 ε_+ 项外，大多数项的推导都是平凡的。我们稍后再讨论这个问题，为方便起见，我们先直接写下 $\partial_i m$ 而非 m 的全变分。根据式 (112)，我们得到

$$\delta\partial_i m = \xi^\varnothing\partial_\varnothing\partial_i m + \xi^j\left(t\right)\partial_j\partial_i m + {\lambda_i}^j\partial_j m - {\lambda^m}_n x^n\partial_m\partial_i m + \partial_i\sigma\left(x^\upsilon\right) - \dot{\xi}^i\left(t\right)$$

$$-\frac{1}{2}\bar{\varepsilon}_+\gamma_i\psi_{\varnothing-} \tag{113}$$

Note that the terms proportional to the local boost parameters $\lambda^i\left(x^\upsilon\right)$ have cancelled out.

注意，正比于局域 boost 参数 $\lambda^i\left(x^\upsilon\right)$ 的项已经抵消了。

We now partially gauge fix the central charge transformations by putting

我们现在通过设定对中心荷变换进行部分规范固定

$$m\left(x^\upsilon\right) = 0. \tag{114}$$

We thus obtain

由此我们得到

$$\partial_i\sigma\left(x^\upsilon\right) = \dot{\xi}^i\left(t\right) + \frac{1}{2}\bar{\varepsilon}_+\gamma_i\psi_{\varnothing-}\left(x^\upsilon\right), \tag{115}$$

which is sufficient to calculate the transformation rule of $\partial_i m_\varnothing$. After this gauge fixing, taking into account all the compensating transformations (see Table 3) and the restriction (112) with $m = 0$ substituted, we find the following transformation rules for the remaining independent fields:

这足以计算 $\partial_i m_\varnothing$ 的变换规则。完成本次规范固定后，考虑所有补偿变换 (见表 3) 以及代入 $m=0$ 后的约束式 (112)，我们得到剩余独立场的变换规则如下:

$$\delta\tau_i = \xi^\varnothing\partial_\varnothing\tau_i + \xi^j(t)\partial_j\tau_i - \dot{\xi}^i(t) + \lambda_{ij}\tau^j - \lambda^k{}_l x^l\partial_k\tau_i - \lambda_i(x^\nu) - \frac{1}{2}\bar{\varepsilon}_+\gamma_i\psi_{\varnothing-},$$

$$\delta\partial_i m_\varnothing = \xi^\varnothing\partial_\varnothing\partial_i m_\varnothing + \xi^j(t)\partial_j\partial_i m_\varnothing + \ddot{\xi}^i(t) - \dot{\xi}^j(t)\partial_i\tau_j + \lambda_i{}^j\partial_j m_\varnothing$$

$$-\lambda^m{}_n x^n\partial_m\partial_i m_\varnothing - \partial_i\left(\lambda^j(x^\upsilon)\tau_j\right) + \bar{\varepsilon}_-(t)\gamma^0\partial_i\psi_{\varnothing-} + \frac{1}{2}\partial_i\left(\omega^a\bar{\varepsilon}_+\gamma_a\psi_{\varnothing-}\right)$$

$$+\frac{1}{2}\bar{\varepsilon}_+\gamma_i\dot{\psi}_{\varnothing-}, \tag{116}$$

$$\delta\psi_{\varnothing-} = \xi^\varnothing\partial_\varnothing\psi_{\varnothing-} + \xi^i(t)\partial_i\psi_{\varnothing-} - \lambda^i{}_j x^j\partial_i\psi_{\varnothing-} + \frac{1}{4}\lambda^{ab}\gamma_{ab}\psi_{\varnothing-}$$

$$+\dot{\varepsilon}_-(t) + \frac{1}{2}\left(\omega_\varnothing{}^a - \dot{\omega}^a\right)\gamma_{a0}\varepsilon_+.$$

Table 3 This table indicates the gauge fixing conditions and corresponding compensating transformations that lead to 3*D*NC supergravity from the viewpoint of a Galilean observer. We have also included the restrictions that follow from the fact that the spin connection field $\omega_\mu{}^{ab}$ is dependent. At the bottom of the table, we have summarized the expressions of the nonzero remaining gauge fields in terms of the Newton potential $\Phi(x^\upsilon)$ and its supersymmetric partner $\chi(x^\upsilon)$, which is related to $\Psi(x^\upsilon)$ via (128)

表 3 本表给出了从伽利略观测者视角得到 3*D*NC 超引力的规范固定条件与对应补偿变换。我们还列入了由自旋联络场 $\omega_\mu{}^{ab}$ 是依赖场这一事实导出的约束。在表底，我们汇总了非零剩余规范场用牛顿势 $\Phi(x^\upsilon)$ 及其超对称伙伴 $\chi(x^\upsilon)$ 表示的表达式，其中 $\chi(x^\upsilon)$ 通过 (128) 式与 $\Psi(x^\upsilon)$ 关联

Gauge condition/Restriction	Compensating transformation
$\tau_\mu(x^\nu) = \delta_\mu\varnothing$	$\xi^\varnothing(x^\upsilon) = \xi^\varnothing$
$\omega_\mu{}^{ab}(x^\nu) = 0$	$\lambda^{ab}(x^\nu) = \lambda^{ab}$
$\psi_{\mu+}(x^\nu) = 0$	$\varepsilon_+(x^\nu) = \varepsilon_+$
$e_i{}^a(x^\nu) = \delta_i{}^a$	$\xi^i(x^\upsilon) = \xi^i(t) - \lambda^i{}_j x^j$
$\psi_{i-}(x^\nu) = 0$	$\varepsilon_-(x^\nu) = \varepsilon_-(t) - \frac{1}{2}\omega^a(x^\nu)\gamma_{a0}\varepsilon_+$
$\tau_i(x^\upsilon) + m_i(x^\upsilon) = \partial_i m(x^\upsilon)$	-
$m(x^\upsilon) = 0$	$\partial_i\sigma(x^\upsilon) = \dot{\xi}^i(t) + \frac{1}{2}\bar{\varepsilon}_+\gamma_i\psi_{\varnothing-}(x^\upsilon)$
$\tau^a(x^\upsilon) = 0$	$\lambda^i(x^\upsilon) = -\ddot{\xi}^i(t) - \frac{1}{2}\bar{\varepsilon}_+\gamma_i\psi_{\varnothing-}(x^\upsilon)$
$m_\varnothing(x^\upsilon) = \Phi(x^\upsilon)$, $\omega_\varnothing{}^a(x^\upsilon) = -\partial^a\Phi(x^\upsilon)$	$\psi_{\varnothing-}(x^\upsilon) = \Psi(x^\upsilon)$

Note that $\omega_\varnothing{}^a$ and ω^a depend on the fields $\tau_i, m_\varnothing$. Using expression (69) for the dependent boost gauge field $\omega_\mu{}^a$, one can calculate that

注意 $\omega_\varnothing{}^a$ 和 ω^a 依赖于场 $\tau_i, m_\varnothing$ 。利用依赖 boost 规范场 $\omega_\mu{}^a$ 的表达式 (69)，可以计算得到

$$\omega_i{}^a \equiv \partial_i\omega^a = -\partial_i\tau^a \;\rightarrow\; \omega^a = -\tau^a, \tag{117}$$

$$\omega_{\varnothing}{}^{a} = -\dot{\tau}^{a} - \partial_a\left(m_{\varnothing} - \frac{1}{2}\tau^i\tau^i\right). \tag{118}$$

As a final step, we now fix the local boost transformations by imposing

作为最后一步，我们现在通过施加条件固定局域 boost 变换

$$\tau^i\left(x^{\upsilon}\right) = 0, \tag{119}$$

which leads to the following compensating transformations:

由此得到如下补偿变换:

$$\lambda^i\left(x^{\upsilon}\right) = -\dot{\xi}^i\left(t\right) - \frac{1}{2}\bar{\varepsilon}_+\gamma_i\psi_{\varnothing-}\left(x^{\upsilon}\right). \tag{120}$$

One now finds that

不难发现

$$\omega^a = 0,\ \omega_{\varnothing}{}^{a} = -\partial^a m_{\varnothing} \equiv -\partial^a\Phi, \tag{121}$$

where Φ is the Newton potential. In terms of the ’ Newton force’ Φ_i and its supersymmetric partner Ψ defined by

其中 Φ 是牛顿势。用如下定义的“牛顿力”Φ_i 及其超对称伙伴 Ψ 表示

$$\Phi_i = \partial_i\Phi,\ \Psi = \psi_{\varnothing-}, \tag{122}$$

one thus obtains the following transformation rules:

由此可以得到如下变换规则:

$$\delta\Phi_i = \xi^{\varnothing}\partial_{\varnothing}\Phi_i + \xi^j\left(t\right)\partial_j\Phi_i + \ddot{\xi}^i\left(t\right) + \lambda_i{}^j\Phi_j - \lambda^m{}_n x^n\partial_m\Phi_i + \bar{\varepsilon}_-\left(t\right)\gamma^0\partial_i\Psi$$

$$+\frac{1}{2}\bar{\varepsilon}_+\gamma_i\dot{\Psi} \tag{123}$$

$$\delta\Psi = \xi^{\varnothing}\partial_{\varnothing}\Psi + \xi^i\left(t\right)\partial_i\Psi - \lambda^i{}_j x^j\partial_i\Psi + \frac{1}{4}\lambda^{ab}\gamma_{ab}\Psi + \dot{\varepsilon}_-\left(t\right) - \frac{1}{2}\Phi^i\gamma_{i0}\varepsilon_+.$$

(124)

Note that the central charge transformations only act on the Newton potential, not on the Newton force.

注意中心荷变换仅作用于牛顿势，不作用于牛顿力。

Determining the transformation rule of the Newton potential Φ is nontrivial, due to the fact that the last term of (123) cannot be manifestly written as a ∂_i -derivative. The above transformation rules are consistent with the integrability condition

确定牛顿势 Φ 的变换规则并非易事，因为 (123) 的最后一项无法直接写为 ∂_i 导数的形式。上述变换规则与可积性条件一致

$$\partial_{[i}\Phi_{j]}\left(x^{\nu}\right)=0, \tag{125}$$

by virtue of the fermionic equations of motion given in Eqs. (76) and (78) which, after gauge fixing, take on the form

这得益于 (76) 和 (78) 给出的费米子运动方程，规范固定后这些方程取如下形式

$$\gamma^{i}\partial_{i}\Psi\left(x^{\nu}\right)=0 \;\Leftrightarrow\; \partial_{[i}\gamma_{j]}\Psi\left(x^{\nu}\right)=0. \tag{126}$$

Under supersymmetry, these fermionic equations of motion lead to the following bosonic equation of motion:

在超对称变换下，这些费米子运动方程导出如下玻色子运动方程:

$$\partial^{i}\Phi_{i}\left(x^{\nu}\right)=0. \tag{127}$$

The same bosonic equation of motion also follows from the last equation in (78) after gauge fixing.

同一个玻色子运动方程也可由规范固定后 (78) 的最后一个方程得到。

In order to obtain transformation rules for the Newton potential Φ and its fermionic superpartner, we need to solve the fermionic equation of motion/constraint (126). The second form of this constraint makes it clear that the equations of motion are solved by a spinor χ , which obeys:

为了得到牛顿势 Φ 及其费米超伙伴的变换规则，我们需要求解费米子运动方程/约束 (126)。该约束的第二种形式清楚表明，运动方程可由满足下式的旋量 χ 解出:

$$\gamma_{i}\Psi=\partial_{i}\chi \tag{128}$$

Note that this only determines χ up to a purely time-dependent shift. From (128), it follows that χ obeys the constraint:

请注意，这仅能在相差一个纯时间依赖平移的范围内确定 χ 。由式 (128) 可得，χ 满足如下约束:

$$\gamma^{1}\partial_{1}\chi=\gamma^{2}\partial_{2}\chi \tag{129}$$

Ψ can thus be expressed in terms of χ in a number of equivalent ways:

因此 Ψ 可以通过多种等价方式用 χ 表示:

$$\Psi = \gamma^1\partial_1\chi = \gamma^2\partial_2\chi = \frac{1}{2}\gamma^i\partial_i\chi. \tag{130}$$

It is now possible to determine the transformation rule of Φ by rewriting $\delta\Phi_i$ as a ∂_i -derivative:

现在我们可以通过将 $\delta\Phi_i$ 重写为 ∂_i 导数来确定 Φ 的变换规则:

$$\delta\Phi_i = \partial_i\left(\delta\Phi\right). \tag{131}$$

The resulting transformation rule for the Newton potential is

牛顿势最终得到的变换规则为

$$\delta\Phi = \xi^{\varnothing}\partial_{\varnothing}\Phi + \xi^i\left(t\right)\partial_i\Phi + \ddot{\xi}^i\left(t\right)x^i - \lambda^m{}_n x^n\partial_m\Phi + \frac{1}{2}\bar{\varepsilon}_-\left(t\right)\gamma^{0i}\partial_i\chi + \frac{1}{2}\bar{\varepsilon}_+\dot{\chi} + \sigma\left(t\right).$$

(132)

Note that we have allowed for an arbitrary time-dependent shift $\sigma\left(t\right)$ in the transformation rule, whose origin stems from the fact that $\Phi_i = \partial_i\Phi$ only determines Φ up to an arbitrary time-dependent shift.

请注意，我们在变换规则中允许 $\sigma\left(t\right)$ 存在任意时间依赖平移，其根源在于 $\Phi_i = \partial_i\Phi$ 仅能在相差一个任意时间依赖平移的范围内确定 Φ 。

To determine the transformation rule of χ , we try to rewrite $\gamma_i\delta\Psi$ as a ∂_i - derivative:

为了确定 χ 的变换规则，我们尝试将 $\gamma_i\delta\Psi$ 重写为 ∂_i 导数:

$$\gamma_i\delta\Psi = \partial_i\left(\delta\chi\right). \tag{133}$$

Most of the terms in $\gamma_i\delta\Psi$ can be straightforwardly written as a ∂_i -derivative. Only for the ε_+ transformation, the argument is a bit subtle. We thus focus on the terms in $\gamma_i\delta\Psi$, given by

$\gamma_i\delta\Psi$ 中的大多数项都可以直接写成 ∂_i 导数。仅 ε_+ 变换的推导过程稍显微妙，因此我们重点讨论 $\gamma_i\delta\Psi$ 中由下式给出的项:

$$-\frac{1}{2}\gamma_i\Phi^j\gamma_{j0}\varepsilon_+ = -\frac{1}{2}\gamma_i\partial^j\Phi\gamma_{j0}\varepsilon_+ = -\frac{1}{2}\partial^j\Phi\gamma_{ij0}\varepsilon_+ - \frac{1}{2}\partial_i\Phi\gamma_0\varepsilon_+. \tag{134}$$

The last term is already in the desired form. To rewrite the first term in the proper form, we note that the Newton potential Φ can be dualized to a ’ dual Newton potential’ Ξ via

最后一项已经符合要求的形式。为了将第一项改写为正确形式，我们注意到牛顿势 Φ 可以通过下式对偶化为“对偶牛顿势” Ξ :

$$\partial_i\Phi = \varepsilon_{ij}\partial^j\Xi,\ \partial_i\Xi = -\varepsilon_{ij}\partial^j\Phi. \tag{135}$$

Using the convention that $\gamma_{ij0} = \varepsilon_{0ij} = \varepsilon_{ij}$, we then get

采用 $\gamma_{ij0} = \varepsilon_{0ij} = \varepsilon_{ij}$ 的约定，我们可得

$$-\frac{1}{2}\gamma_i \Phi^j \gamma_{j0}\varepsilon_+ = \frac{1}{2}\partial_i \Xi \varepsilon_+ - \frac{1}{2}\partial_i \Phi \gamma_0 \varepsilon_+. \tag{136}$$

One thus obtains the following transformation rule for χ , which includes the dual Newton potential Ξ :

由此我们得到 χ 的如下变换规则，其中包含对偶牛顿势 Ξ :

$$\delta\chi = \xi^{\varnothing}\partial_{\varnothing}\chi + \xi^i(t)\partial_i\chi - \lambda^m{}_n x^n \partial_m \chi + \frac{1}{4}\lambda^{mn}\gamma_{mn}\chi$$

$$+x^i\gamma_i\dot{\varepsilon}_-(t) + \frac{1}{2}\Xi\varepsilon_+ - \frac{1}{2}\Phi\gamma_0\varepsilon_+ + \eta(t). \tag{137}$$

Note that we have again allowed for a purely time-dependent shift $\eta(t)$, whose origin lies in the fact that (128) only determines χ up to a purely time-dependent shift.

请注意，我们再次允许 $\eta(t)$ 存在一个纯时间依赖平移，其根源在于式 (128) 仅能在相差一个纯时间依赖平移的范围内确定 χ 。

In order to calculate the algebra on Φ and χ , we also need the transformation rule of the dual potential Ξ . This rule is determined by dualizing the transformation rule of Φ :

为了计算 Φ 和 χ 上的代数，我们还需要对偶势 Ξ 的变换规则。该规则通过对偶化 Φ 的变换规则得到:

$$\partial_i(\delta\Xi) = -\varepsilon_{ij}\partial^j(\delta\Phi). \tag{138}$$

By repeatedly using (128) and (135), we obtain:

反复利用式 (128) 和 (135)，我们得到:

$$\delta\Xi = \xi^{\varnothing}\partial_{\varnothing}\Xi + \xi^i(t)\partial_i\Xi + \ddot{\xi}^i(t)\varepsilon_{ij}x^j - \lambda^m{}_n x^n\partial_m\Xi$$

$$+\frac{1}{2}\bar{\varepsilon}_-(t)\gamma^i\partial_i\chi - \frac{1}{2}\bar{\varepsilon}_+\gamma_0\dot{\chi} + \tau(t), \tag{139}$$

where we again allowed for a purely time-dependent shift $\tau(t)$.

其中我们再次允许 $\tau(t)$ 存在一个纯时间依赖平移。

The algebra then closes on Φ and χ , using (128),(129) and (135). One finds the following nonzero commutators between the fermionic symmetries:

利用式 (128)、(129) 和 (135)，代数最终在 Φ 和 χ 上闭合，我们得到费米对称性之间如下非对易子:

$$\left[\delta_{\varepsilon_{1-}(t)}, \delta_{\varepsilon_{2-}(t)}\right] = \delta_{\sigma(t)}\left(\frac{\mathrm{d}}{\mathrm{dt}}\left(\bar{\varepsilon}_{2-}(t)\gamma^0\varepsilon_{1-}(t)\right)\right),$$

$$\left[\delta_{\varepsilon_{1+}}, \delta_{\varepsilon_{2+}}\right] = \delta_{\xi^{\varnothing}}\left(\frac{1}{2}\left(\bar{\varepsilon}_{2+}\gamma^0\varepsilon_{1+}\right)\right),$$

$$\left[\delta_{\varepsilon_{+}}, \delta_{\varepsilon_{-}(t)}\right] = \delta_{\xi^i(t)}\left(\frac{1}{2}\left(\bar{\varepsilon}_{-}(t)\gamma^i\varepsilon_{+}\right)\right),$$

$$\left[\delta_{\eta(t)}, \delta_{\varepsilon_{+}}\right] = \delta_{\sigma(t)}\left(\frac{1}{2}\left(\bar{\varepsilon}_{+}\dot{\eta}(t)\right)\right). \tag{140}$$

The nonzero commutators between the bosonic and fermionic symmetries are given by:

玻色对称性与费米对称性之间的非零对易子由下式给出:

$$\left[\delta_{\xi^i(t)}, \delta_{\varepsilon_{+}}\right] = \delta_{\varepsilon_{-}(t)}\left(\frac{1}{2}\dot{\xi}^i(t)\gamma_{0i}\varepsilon_{+}\right),\ \left[\delta_{\lambda^{ij}}, \delta_{\varepsilon_{+}}\right] = \delta_{\varepsilon_{+}}\left(-\frac{1}{4}\lambda^{ij}\gamma_{ij}\varepsilon_{+}\right),$$

$$\left[\delta_{\xi^{\varnothing}}, \delta_{\varepsilon_{-}(t)}\right] = \delta_{\varepsilon_{-}(t)}\left(-\xi^{\varnothing}\dot{\varepsilon}_{-}(t)\right),\ \left[\delta_{\xi^i(t)}, \delta_{\varepsilon_{-}(t)}\right] = \delta_{\eta(t)}\left(-\xi^i(t)\gamma_i\dot{\varepsilon}_{-}(t)\right),$$

$$\left[\delta_{\lambda^{ij}}, \delta_{\varepsilon_{-}(t)}\right] = \delta_{\varepsilon_{-}(t)}\left(-\frac{1}{4}\lambda^{ij}\gamma_{ij}\varepsilon_{-}(t)\right), \left[\delta_{\sigma(t)}, \delta_{\varepsilon_{+}}\right] = \delta_{\eta(t)}\left(\frac{1}{2}\left(\sigma(t)\gamma^0\varepsilon_{+}\right)\right),$$

$$\left[\delta_{\xi^{\varnothing}}, \delta_{\eta(t)}\right] = \delta_{\eta(t)}\left(-\xi^{\varnothing}\dot{\eta}(t)\right),\ \left[\delta_{\lambda^{ij}}, \delta_{\eta(t)}\right] = \delta_{\eta(t)}\left(-\frac{1}{4}\lambda^{ij}\gamma_{ij}\eta(t)\right).$$

(141)

The bosonic commutators are not changed with respect to the purely bosonic case and are given by (102).

玻色对易子与纯玻色情形相比没有发生变化，其形式由式 (102) 给出。

This finishes our discussion of the $\mathcal{N} = 2$ Galilean supergravity theory. Like in the bosonic case, we have summarized all gauge fixing conditions and resulting compensating transformations in Table 4. In the next section, we will discuss another type of $3D$ non-Lorentzian supergravity theories that are of a Chern-Simons form based on a (super-)Lie algebra.

至此我们对 $\mathcal{N} = 2$ 伽利略超引力理论的讨论就完成了。与玻色情形相同，我们已将所有规范固定条件和由此得到的补偿变换汇总在表 4 中。下一节我们将讨论另一类基于 (超) 李代数、具有陈-西蒙斯形式的 $3D$ 非洛伦兹超引力理论。

Non-Relativistic 3D Chern-Simons Supergravity and Lie Algebra Expansions

非相对论三维陈-西蒙斯超引力与李代数扩张

Above, we discussed the supergravity version of $3D$NC gravity, the diffeomorphism covariant formulation of Newtonian gravity. Although we focused on the construction of this theory via a NR limit, we mentioned that it can also be constructed as a gauging of the supersymmetric extension of the Bargmann algebra, given in (58) [10]. This gauging perspective highlights an important structural difference between 3DNC (super)gravity and its relativistic Einstein-Hilbert counterpart: while the latter can be written as a Chern-Simons theory [48-50], the former cannot. This can be seen from the fact that the Bargmann algebra cannot be equipped with an invariant bilinear form that is required to construct a Chern-Simons action.

上文我们讨论了 3DNC 引力的超引力版本，即牛顿引力的微分同胚协变表述。尽管我们重点讨论了通过非相对论极限构造该理论的方法，但也提到它也可以通过对 (58) 中给出的 Bargmann 代数超对称扩展做定域规范变换来构造 [10]。这种规范视角凸显了 3DNC (超) 引力与其相对论对应理论爱因斯坦-希尔伯特引力之间一个重要的结构差异: 后者可以写作陈-西蒙斯理论 [48-50]，而前者不行。这一点可以从以下事实看出:Bargmann 代数不存在构造陈-西蒙斯作用量所需的不变双线性型。

It is natural to ask whether there also exist NR 3D supergravity theories that admit a first-order Chern-Simons formulation. A hint that this is possible is provided by the existence of a purely bosonic 3DNR gravity theory in Chern-Simons form [19]. This relies on the fact that the 3D Bargmann algebra admits an extra central extension. The nonzero commutation relations of this centrally extended Bargmann algebra are given by:

我们自然会问: 是否存在允许一阶陈-西蒙斯表述的非相对论 3D 超引力理论？纯玻色子 3DNR 引力存在陈-西蒙斯形式的理论 [19] 提示我们这是可能的。这依赖于一个事实: 3D Bargmann 代数允许额外中心扩张。这个中心扩张 Bargmann 代数的非零对易关系如下:

$$[H, G_a] = -\varepsilon_{ab}P_b,\ [J, G_a] = -\varepsilon_{ab}G_b,\ [J, P_a] = -\varepsilon_{ab}P_b,$$

$$[G_a, G_b] = \varepsilon_{ab}S,\ [G_a, P_b] = \varepsilon_{ab}M. \tag{142}$$

Here, S denotes the extra central extension, and we have replaced J_{ab} by $\varepsilon_{ab}J$ and G_a by $\varepsilon_{ab}G_b$ in the bosonic part of (58). In contrast to the Bargmann algebra, the extended Bargmann algebra (142) can be equipped with a trace, i.e., a nondegenerate, invariant bilinear form. Explicitly, this trace takes on the following nonzero values when evaluated on the generators of the extended Bargmann algebra [19]:

此处 S 表示额外中心扩张，我们已将 (58) 玻色子部分中的 J_{ab} 替换为 $\varepsilon_{ab}J$，将 G_a 替换为 $\varepsilon_{ab}G_b$。与原 Bargmann 代数不同，扩张后的 Bargmann 代数 (142) 可以配备一个迹，即非退化不变双线性型。具体来说，该迹作用在扩张 Bargmann 代数的生成元上得到的非零值如下 [19]:

$$\mathrm{tr}\,(G_a P_b) = \delta_{ab},\ \mathrm{tr}\,(HS) = \mathrm{tr}\,(MJ) = -1. \tag{143}$$

One can then work out the generic form of the Chern-Simons action

接下来我们可以推导出陈-西蒙斯作用量的一般形式

$$S = \frac{k}{4\pi}\int \mathrm{tr}\Big(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\Big), \tag{144}$$

with k the Chern-Simons coupling constant and $A = A_\mu \mathrm{d}x^\mu$ a Lie algebra-valued gauge field, for the algebra (142) and trace (143). Parametrizing the gauge field A as follows:

其中 k 是陈-西蒙斯耦合常数，$A = A_\mu dx^\mu$ 是李代数取值的规范场，对应代数 (142) 和迹 (143)。我们对规范场 A 做如下参数化:

$$A_\mu = \tau_\mu H + e_\mu{}^a P_a + \omega_\mu J + \omega_\mu{}^a G_a + m_\mu M + s_\mu S. \tag{145}$$

this leads to the following action [19]:

由此得到如下作用量 [19]:

$$S = \frac{k}{4\pi}\int d^3x \left(\varepsilon^{\mu\nu\rho} e_\mu{}^a R_{\nu\rho a}(G) - \varepsilon^{\mu\nu\rho} m_\mu R_{\nu\rho}(J) - \varepsilon^{\mu\nu\rho}\tau_\mu R_{\nu\rho}(S)\right),$$

(146)

where

其中

$$R_{\mu\nu}{}^a(G) \equiv 2\partial_{[\mu}\omega_{\nu]}{}^a + 2\varepsilon^{ab}\omega_{[\mu}\omega_{\nu]b},\ R_{\mu\nu}(J) \equiv 2\partial_{[\mu}\omega_{\nu]},$$

$$R_{\mu\nu}(S) \equiv 2\partial_{[\mu}s_{\nu]} + \varepsilon^{ab}\omega_{[\mu a}\omega_{\nu]b}. \tag{147}$$

Upon examining the gauge transformation rule of A_μ , one finds that the fields τ_μ , $e_\mu{}^a$ and m_μ transform under local boosts G_a , spatial rotations J and the central charge M as in (11), showing that they can be identified as the time-like and spatial vielbein and the central charge gauge field of NC geometry. The fields ω_μ and $\omega_\mu{}^a$ can likewise be interpreted as spin connections for local spatial rotations and boosts, respectively. Varying (146) with respect to ω_μ and $\omega_\mu{}^a$ yields:

考察 A_μ 的规范变换规则后可以发现，场 τ_μ 、$e_\mu{}^a$ 和 m_μ 在局域 boost G_a 、空间转动 J 和中心荷 M 下的变换与 (11) 一致，说明它们可以对应非对易几何的类时 vielbein、空间 vielbein 和中心荷规范场。场 ω_μ 和 $\omega_\mu{}^a$ 则可以分别解释为局域空间转动和 boost 的自旋联络。对 ω_μ 和 $\omega_\mu{}^a$ 变分 (146) 可得:

$$R_{\mu\nu}{}^a(P) \equiv 2\partial_{[\mu}e_{\nu]}{}^a + 2\varepsilon^{ab}\omega_{[\mu}e_{\nu]b} - 2\varepsilon^{ab}\omega_{[\mu b}\tau_{\nu]} = 0,$$

$$R_{\mu\nu}(M) \equiv 2\partial_{[\mu}m_{\nu]} + 2\varepsilon^{ab}\omega_{[\mu a}e_{\nu]b} = 0. \tag{148}$$

These are the conventional constraints (22) (adapted to the notation and conventions of (142)) that can be used to express the spin connections ω_μ and $\omega_\mu{}^a$ as dependent fields. Note that the torsion tensors $T_{\mu\nu}{}^a$ and $T^{(m)}_{\mu\nu}$ are zero here, as is the intrinsic torsion, since varying (146) with respect to s_μ yields the absolute time constraint (29). Extremizing the action (146) thus yields torsionless NC geometries. The other equations of motion that follow from varying (146) with respect to $\tau_\mu, e_\mu{}^a$ and m_μ are obtained by setting the curvatures (147) equal to zero. The first two of these imply that the NC geometries involved have vanishing Riemann curvature. This is analogous to $3D$ relativistic gravity, so that the Chern-Simons action (146) can be viewed as a first-order formulation of a NR gravity theory in three dimensions. This theory is called 'extended Bargmann gravity'.

这些就是常规约束 (22)(适配 (142) 的记号与约定)，可用于将自旋联络 ω_μ 和 $\omega_\mu{}^a$ 表示为依赖场。注意此处挠率张量 $T_{\mu\nu}{}^a$ 和 $T_{\mu\nu}^{(m)}$ 为零，内蕴挠率也为零，因为对 s_μ 变分 (146) 会得到绝对时间约束 (29)。因此对作用量 (146) 做极值化得到的是无挠非对易几何。对 $\tau_\mu, e_\mu{}^a$ 和 m_μ 变分 (146) 得到其余运动方程，要求曲率 (147) 为零。前两个方程说明所得非对易几何的黎曼曲率为零。这与 $3D$ 相对论引力类似，因此陈-西蒙斯作用量 (146) 可以看作三维非相对论引力理论的一阶表述，该理论被称为“扩张 Bargmann 引力”。

A supergravity generalization of extended Bargmann gravity can be constructed, if one can find a superalgebra with a non-degenerate invariant supertrace that contains (142) as a bosonic subalgebra. An example of such a superalgebra was found in [18] by trial and error. It extends (142) with three fermionic generators Q^+, Q^- and R (that are Majorana spinors), and its nonzero (anti-)commutation relations are given by (142), as well as the following ones:

若能找到一个具有非退化不变超迹的超代数，且它以 (142) 为玻色子子代数，便可构造扩展 Bargmann 引力的超引力推广。这类超代数的一个例子已在文献 [18] 中通过试错法找到。它在 (142) 的基础上额外扩展了三个费米生成元 Q^+, Q^- 和 R (均为马约拉纳旋量)，其非零 (反) 对易关系除 (142) 外还包括如下:

$$[J, Q^\pm] = -\frac{1}{2}\gamma_0 Q^\pm,\ [J, R] = -\frac{1}{2}\gamma_0 R,\ [G_a, Q^+] = -\frac{1}{2}\gamma_a Q^-,$$

$$[G_a, Q^-] = -\frac{1}{2}\gamma_a R,\ [S, Q^+] = -\frac{1}{2}\gamma_0 R,$$

$$\left\{Q^+_\alpha, Q^+_\beta\right\} = \left(\gamma_0 C^{-1}\right)_{\alpha\beta} H,\ \left\{Q^+_\alpha, Q^-_\beta\right\} = -\left(\gamma^a C^{-1}\right)_{\alpha\beta} P_a,$$

$$\left\{Q^-_\alpha, Q^-_\beta\right\} = \left(\gamma_0 C^{-1}\right)_{\alpha\beta} M,\ \left\{Q^+_\alpha, R_\beta\right\} = \left(\gamma_0 C^{-1}\right)_{\alpha\beta} M. \tag{149}$$

This ’extended Bargmann superalgebra’ can be equipped with an invariant super-trace that is given by (143), extended with

这种“扩展 Bargmann 超代数”可配备一个不变超迹，该超迹由 (143) 给出，并额外扩展了

$$\operatorname{tr}\left(Q^+_\alpha R_\beta\right) = 2\left(C^{-1}\right)_{\alpha\beta},\ \operatorname{tr}\left(Q^-_\alpha Q^-_\beta\right) = 2\left(C^{-1}\right)_{\alpha\beta}. \tag{150}$$

Introducing the gauge field

引入规范场

$$A_\mu = \tau_\mu H + e_\mu{}^a P_a + \omega_\mu J + \omega_\mu{}^a G_a + m_\mu M + s_\mu S$$

$$+\overline{\psi}^+_\mu Q^+ + \overline{\psi}^-_\mu Q^- + \overline{\rho}_\mu R \tag{151}$$

where $\psi^\pm_\mu, \rho_\mu$ are Majorana, the Chern-Simons action for the superalgebra (142), (149) is found to be given by

其中 $\psi_\mu^\pm, \rho_\mu$ 为马约拉纳旋量，可得对应于超代数 (142)、(149) 的陈-西蒙斯作用量为

$$S = \frac{k}{4\pi}\int \mathrm{d}^3x \left(\varepsilon^{\mu\upsilon\rho} e_\mu{}^a R_{\upsilon\rho a}(G) - \varepsilon^{\mu\upsilon\rho} m_\mu R_{\upsilon\rho}(J) - \varepsilon^{\mu\upsilon\rho}\tau_\mu R_{\upsilon\rho}(S)\right.$$

$$\left. + \varepsilon^{\mu\upsilon\rho}\overline{\psi}^+_\mu \hat{\rho}_{\upsilon\rho} + \varepsilon^{\mu\upsilon\rho}\overline{\rho}_\mu \widehat{\psi}^+_{\upsilon\rho} + \varepsilon^{\mu\upsilon\rho}\overline{\psi}^-_\mu \widehat{\psi}^-_{\upsilon\rho}\right). \tag{152}$$

Here, the supercovariant curvatures are given by

此处，超协变曲率由下式给出

$$\widehat{\psi}^+_{\mu\nu} = 2\partial_{[\mu}\psi^+_{\nu]} + \omega_{[\mu}\gamma_0\psi^+_{\nu]},$$

$$\widehat{\psi}^-_{\mu\upsilon} = 2\partial_{[\mu}\psi^-_{\upsilon]} + \omega_{[\mu}\gamma_0\psi^-_{\upsilon]} + \omega_{[\mu}{}^a\gamma_a\psi^+_{\upsilon]},$$

$$\hat{\rho}_{\mu\upsilon} = 2\partial_{[\mu}\rho_{\upsilon]} + \omega_{[\mu}\gamma_0\rho_{\upsilon]} + \omega_{[\mu}{}^a\gamma_a\psi^-_{\upsilon]} + s_{[\mu}\gamma_0\psi^+_{\upsilon]}. \tag{153}$$

Denoting the parameters of the local $Q^\pm$ and R transformations by $\varepsilon^\pm$ and η , respectively, one finds that the action (152) is invariant under the following supersymmetry transformation rules:

将局域 $Q^\pm$ 变换和局域 R 变换的参数分别记为 $\varepsilon^\pm$ 和 η ，可得作用量 (152) 在下述超对称变换规则下不变:

$$\delta\tau_\mu = -\overline{\varepsilon}^+\gamma_0\psi^+_\mu$$

$$\delta e_\mu{}^a = \overline{\varepsilon}^+\gamma^a\psi^-_\mu + \overline{\varepsilon}^-\gamma^a\psi^+_\mu,$$

$$\delta m_\mu = -\overline{\varepsilon}^-\gamma_0\psi^-_\mu - \overline{\varepsilon}^+\gamma_0\rho_\mu - \overline{\eta}\gamma_0\psi^+_\mu,$$

$$\delta\psi^+_\mu = \partial_\mu\varepsilon^+ + \frac{1}{2}\omega_\mu\gamma_0\varepsilon^+,$$

$$\delta\psi^-_\mu = \partial_\mu\varepsilon^- + \frac{1}{2}\omega_\mu\gamma_0\varepsilon^- + \frac{1}{2}\omega_\mu{}^a\gamma_a\varepsilon^+,$$

$$\delta\rho_\mu = \partial_\mu\eta + \frac{1}{2}\omega_\mu\gamma_0\eta + \frac{1}{2}\omega_\mu{}^a\gamma_a\varepsilon^- + \frac{1}{2}s_\mu\gamma_0\varepsilon^+. \tag{154}$$

Unlike the supersymmetric version of the Bargmann algebra (58), the extended Bargmann superalgebra (142), (149) does not correspond to an Inönü-Wigner contraction of a relativistic superalgebra (This statement only holds for the full superalgebra. The bosonic part (142) can be obtained as an Inönü-Wigner contraction of a direct product of the Poincaré algebra with a two-dimensional abelian algebra [18].). It can however be obtained from the three-dimensional $\mathcal{N} = 2$ super-Poincaré algebra (55) by performing a so-called Lie algebra expansion. Two different Lie algebra expansion procedures have been used in the literature, based on expanding Maurer-Cartan equations or using semigroups, respectively. In what follows, we will show how the extended Bargmann superalgebra corresponds to a Lie algebra expansion, according to the Maurer-Cartan equation method of [51, 52]. We refer to the original literature [53] for an account of semigroup expansions. Let us also note that Lie algebra expansions have not only been used to obtain the extended Bargmann superalgebra but also various generalizations thereof that include, e.g., more generators and/or a nontrivial cosmological constant [24,26,28-30]. They have been used to construct examples of four-dimensional NR superalgebras and related supergravity theories [54]. A procedure to construct off-shell rigid matter multiplets for NR algebras obtained via a Lie algebra expansion has been given in [55].

与 Bargmann 代数的超对称版本 (58) 不同，扩展 Bargmann 超代数 (142)、(149) 并不对应相对论超代数的 Inönü-Wigner 收缩 (该结论仅对完整超代数成立，其玻色子部分 (142) 可由庞加莱代数与二维阿贝尔代数直积的 Inönü-Wigner 收缩得到 [18])。不过，它可以通过所谓的李代数展开从三维 $\mathcal{N}=2$ 超庞加莱代数 (55) 得到。文献中已使用两种不同的李代数展开方法，分别基于展开莫雷-嘉当方程，或利用半群。下文我们将根据文献 [51,52] 的莫雷-嘉当方程方法，说明扩展 Bargmann 超代数如何对应李代数展开。关于半群展开的说明可参阅原始文献 [53]。还需指出，李代数展开不仅用于得到扩展 Bargmann 超代数，还可得到其各类推广，例如包含更多生成元、或带有非平凡宇宙学常数的情况 [24,26,28-30]。这类方法已被用于构造四维非相对论超代数及相关超引力理论的例子 [54]。文献 [55] 已经给出了为李代数展开得到的非相对论代数构造脱壳刚性物质多重态的流程。

Let us first give a general overview of the Lie algebra expansion procedure of [51,52]. Consider a Lie algebra $\mathfrak{g}$, with generators T_α and structure constants $f_{\beta\gamma}^{\alpha}$ $(\alpha, \beta, \gamma = 1, \ldots, \dim(\mathfrak{g}))$. The Lie algebra expansion method of [51, 52] uses that $\mathfrak{g}$ can be specified in a dual manner in terms of a Maurer-Cartan one-form. This is a Lie algebra valued one-form $A^\alpha = A_\mu^\alpha \mathrm{d}x^\mu$, satisfying the so-called Maurer-Cartan equations that state that the covariant curvature two-form of A^α vanishes:

我们先对文献 [51,52] 的李代数展开方法做一个总体概述。给定李代数 $\mathfrak{g}$ ，其生成元为 T_α ，结构常数为 $f_{\beta\gamma}^{\alpha}$ $(\alpha, \beta, \gamma = 1, \ldots, \dim(\mathfrak{g}))$ 。[51, 52] 的李代数展开方法利用了如下性质: $\mathfrak{g}$ 可以通过莫雷-嘉当一元形以对偶方式描述。这是一个李代数值一元形 $A^\alpha = A_\mu^\alpha \mathrm{d}x^\mu$ ，满足所谓的莫雷-嘉当方程，即 A^α 的协变曲率二元形为零:

$$F^\alpha \equiv \mathrm{d}A^\alpha + \frac{1}{2} f_{\beta\gamma}^{\alpha} A^\beta \wedge A^\gamma = 0. \tag{155}$$

In this dual formulation, the Jacobi identities for the structure constants $f_{\beta\gamma}^{\alpha}$ are encoded in the consistency of the Maurer-Cartan equations with $\mathrm{d}^2 = 0$. Conversely, given a set of two-form equations of the form (155) that are consistent with $\mathrm{d}^2 = 0$, one can conclude that the $f_{\beta\gamma}^{\alpha}$ that appear in them satisfy the Jacobi identities and are thus structure constants of a Lie algebra.

在这种对偶表述中，结构常数 $f_{\beta\gamma}^{\alpha}$ 满足的雅可比恒等式，编码在莫雷-嘉当方程对 $\mathrm{d}^2 = 0$ 的自洽性中。反之，若给定一组形如 (155)、且对 $\mathrm{d}^2 = 0$ 自洽的二元形方程，就可以推导出其中出现的 $f_{\beta\gamma}^{\alpha}$ 满足雅可比恒等式，因此是某个李代数的结构常数。

In what follows, we will assume (This assumption is satisfied in most of the applications of Lie algebra expansions to obtain new NR (super)algebras. Note however that the Lie algebra expansion method of [51, 52] is more general and can also be applied to different cases where this assumption does not hold.) that $\mathfrak{g}$ can be decomposed (as a vector space) as $\mathfrak{g} = V_0 \oplus V_1$, such that

在下文中，我们假设 (该假设在李代数展开得到新非相对论 (超) 代数的大多数应用中都成立。但需注意，文献 [51,52] 的李代数展开方法更具一般性，也可应用于该假设不成立的不同情况): 作为向量空间，$\mathfrak{g}$ 可分解为 $\mathfrak{g} = V_0 \oplus V_1$ ，满足

$$[V_0, V_0] \subset V_0, \ [V_0, V_1] \subset V_1, \ [V_1, V_1] \subset V_0. \tag{156}$$

We will split the Lie algebra index α into α_0 to refer to components along V_0 and α_1 to refer to components along V_1 . The Maurer-Cartan equations then take the following form:

我们将李代数指标 α 拆分为 α_0 指代沿 V_0 的分量，α_1 指代沿 V_1 的分量。此时 Maurer-Cartan 方程取如下形式:

$$F^{\alpha_0} = \mathrm{d}A^{\alpha_0} + \frac{1}{2} f^{\alpha_0}_{\beta_0\gamma_0} A^{\beta_0} \wedge A^{\gamma_0} + \frac{1}{2} f^{\alpha_0}_{\beta_1\gamma_1} A^{\beta_1} \wedge A^{\gamma_1} = 0,$$

$$F^{\alpha_1} = \mathrm{d}A^{\alpha_1} + f^{\alpha_1}_{\beta_0\gamma_1} A^{\beta_0} \wedge A^{\gamma_1} = 0. \tag{157}$$

The application of the Lie algebra expansion procedure to $\mathfrak{g}$ proceeds by first considering a formal power series expansion of the Maurer-Cartan one-form A^α in an expansion parameter λ . For the components of A^α along V_0 , this expansion starts at order λ^0 , while for those along V_1 , it starts at order λ :

将李代数展开流程应用于 $\mathfrak{g}$ 时，我们首先对 Maurer-Cartan 一元形式 A^α 在展开参数 λ 下做形式幂级数展开。对于 A^α 沿 V_0 的分量，展开从 λ^0 阶开始；而对于沿 V_1 的分量，展开从 λ 阶开始:

$$A^{\alpha_0} = \sum_{n=0}^{\infty} \lambda^{2n} A^{\alpha_0}_{(2n)} = A^{\alpha_0}_{(0)} + \lambda^2 A^{\alpha_0}_{(2)} + \lambda^4 A^{\alpha_0}_{(4)} + \cdots,$$

$$A^{\alpha_1} = \sum_{n=0}^{\infty} \lambda^{2n+1} A^{\alpha_1}_{(2n+1)} = \lambda A^{\alpha_1}_{(1)} + \lambda^3 A^{\alpha_1}_{(3)} + \lambda^5 A^{\alpha_1}_{(5)} + \cdots. \tag{158}$$

By plugging these expansions in the Maurer-Cartan equations (157), collecting like powers of λ and setting the coefficients of each power of λ equal to zero, one obtains Maurer-Cartan-like equations for an infinite tower of one-forms $A^{\alpha_0}_{(2n)}, A^{\alpha_1}_{(2m+1)}$ $(n, m = 0, 1, 2, \ldots)$. These equations are consistent with $\mathrm{d}^2 = 0$ by virtue of the

将这些展开式代入莫雷尔-嘉当方程 (157)，按 λ 的幂次整理并令 λ 各幂次的系数等于零后，便可得到无穷串一元形式 $A^{\alpha_0}_{(2n)}, A^{\alpha_1}_{(2m+1)}$ $(n, m = 0, 1, 2, \ldots)$ 满足的类莫雷尔-嘉当方程。这些方程相对于 $\mathrm{d}^2 = 0$ 是一致的，这得益于

Jacobi identities for $f^{\alpha}_{\beta\gamma}$, so that they correspond to the Maurer-Cartan equations of an infinite-dimensional Lie algebra. To get a finite-dimensional Lie algebra, one looks for consistent truncations of the expansions (158), i.e., one proposes the expansions:

$f^{\alpha}_{\beta\gamma}$ 的雅可比恒等式，因此它们对应一个无穷维李代数的莫雷尔-嘉当方程。为得到有限维李代数，我们需要对展开式 (158) 寻找一致截断，即设展开式为:

$$A^{\alpha_0} = \sum_{n=0}^{N_0/2} \lambda^{2n} A^{\alpha_0}_{(2n)} = A^{\alpha_0}_{(0)} + \lambda^2 A^{\alpha_0}_{(2)} + \cdots + \lambda^{N_0} A^{\alpha_0}_{(N_0)},$$

$$A^{\alpha_1} = \sum_{n=0}^{(N_1-1)/2} \lambda^{2n+1} A^{\alpha_1}_{(2n+1)} = \lambda A^{\alpha_1}_{(1)} + \lambda^3 A^{\alpha_1}_{(3)} + \cdots + \lambda^{N_1} A^{\alpha_1}_{(N_1)} \tag{159}$$

for even integers N_0 and odd integers N_1 . These lead to the following Maurer-Cartan-like equations:

其中 N_0 为偶整数，N_1 为奇整数。由此可得如下类 Maurer-Cartan 方程:

$$\mathrm{d}A^{\alpha_0}_{(2n_0)} + \frac{1}{2} f^{\alpha_0}_{\beta_0\gamma_0} \sum_{r=0}^{n_0} A^{\beta_0}_{(2r)} \wedge A^{\gamma_0}_{(2(n_0-r))} + \frac{1}{2} f^{\alpha_0}_{\beta_1\gamma_1} \sum_{r=1}^{n_0} A^{\beta_1}_{(2r-1)} \wedge A^{\gamma_1}_{(2(n_0-r)+1)} = 0,$$

$$\mathrm{d}A^{\alpha_1}_{(2n_1+1)} + f^{\alpha_1}_{\beta_0\gamma_1} \sum_{r=0}^{n_1} A^{\beta_0}_{(2r)} \wedge A^{\gamma_1}_{(2(n_1-r)+1)} = 0, \tag{160}$$

with $n_0 = 0, 1, 2, \ldots, N_0/2$ and $n_1 = 0, 1, 2, \ldots, (N_1 - 1)/2$. Consistency with $\mathrm{d}^2 = 0$ is again guaranteed by the Jacobi identities for $f^{\alpha}_{\beta\gamma}$, provided that the above Maurer-Cartan-like equations for $A^{\alpha_0}_{(0)}, A^{\alpha_0}_{(2)}, \ldots, A^{\alpha_0}_{(N_0)}$ do not contain any $A^{\alpha_1}_{(n_1)}$ with $n_1 > N_1$ and that those for $A^{\alpha_1}_{(1)}, A^{\alpha_1}_{(3)}, \ldots, A^{\alpha_1}_{(N_1)}$ do not contain any $A^{\alpha_0}_{(n_0)}$ with $n_0 > N_0$. From (160) and taking into account that N_0 and N_1 are even and odd respectively, one finds that this requires that

其中满足 $n_0 = 0, 1, 2, \ldots, N_0/2$ 和 $n_1 = 0, 1, 2, \ldots, (N_1 - 1)/2$ 。只要上述 $A^{\alpha_0}_{(0)}, A^{\alpha_0}_{(2)}, \ldots, A^{\alpha_0}_{(N_0)}$ 的类 Maurer-Cartan 方程不包含任何满足 $n_1 > N_1$ 的 $A^{\alpha_1}_{(n_1)}$ ，且 $A^{\alpha_1}_{(1)}, A^{\alpha_1}_{(3)}, \ldots, A^{\alpha_1}_{(N_1)}$ 的方程不包含任何满足 $n_0 > N_0$ 的 $A^{\alpha_0}_{(n_0)}$ ，那么利用 $f^{\alpha}_{\beta\gamma}$ 的雅可比恒等式即可再次保证方程与 $\mathrm{d}^2 = 0$ 自洽。结合 (160)，且考虑到 N_0 和 N_1 分别为偶整数和奇整数，我们发现这要求

$$N_1 = N_0 \pm 1 \tag{161}$$

In case this condition is fulfilled, the truncated Maurer-Cartan-like equations (160) correspond to the Maurer-Cartan equations of a Lie algebra, whose structure constants can be read off from the left-hand sides of (160). For $N_0 = 0$ and $N_1 = 1$, this Lie algebra is an Inönü-Wigner contraction of $\mathfrak{g}$. For other allowed values of N_0 and N_1 , it has more generators than $\mathfrak{g}$ and is called a Lie algebra expansion of $\mathfrak{g}$.

若满足该条件，截断的类 Maurer-Cartan 方程 (160) 对应某个李代数的 Maurer-Cartan 方程，其结构常数可从 (160) 的左侧读出。对于 $N_0 = 0$ 和 $N_1 = 1$ ，该李代数是 $\mathfrak{g}$ 的 Inönü-Wigner 收缩。对于 N_0 和 N_1 的其他允许取值，该李代数比 $\mathfrak{g}$ 拥有更多生成元，被称为 $\mathfrak{g}$ 的李代数展开。

To see that the extended Bargmann superalgebra (142), (149) is a Lie algebra expansion of the three-dimensional $\mathcal{N} = 2$ super-Poincaré algebra (55), one then proceeds as follows [33]. Introducing the $\mathcal{N} = 2$ super-Poincaré algebra-valued gauge field (with $J_{\hat{A}} = -(1/2)\,\varepsilon_{\hat{A}}{}^{\hat{B}\hat{C}} M_{\hat{B}\hat{C}}$ (Here, we use the convention that $\varepsilon^{012} = -\varepsilon_{012} = 1$.))

为证明扩展 Bargmann 超代数 (142)、(149) 是三维 $\mathcal{N} = 2$ 超庞加莱代数 (55) 的李代数展开，可按如下步骤进行 [33]。引入取值于 $\mathcal{N} = 2$ 超庞加莱代数的规范场 (其中 $J_{\hat{A}} = -(1/2)\,\varepsilon_{\hat{A}}{}^{\hat{B}\hat{C}} M_{\hat{B}\hat{C}}$ (此处我们约定 $\varepsilon^{012} = -\varepsilon_{012} = 1$.))

$$A_\mu = E_\mu{}^{\hat{A}} P_{\hat{A}} + \Omega_\mu{}^{\hat{A}} J_{\hat{A}} + \bar{\varepsilon}^i Q_i, \tag{162}$$

the Maurer-Cartan equations corresponding to (55) are obtained by setting the supercovariant curvatures

对应 (55) 的 Maurer-Cartan 方程可通过令超协变曲率

$$R_{\mu\nu}{}^{\hat{A}}(P) \equiv 2\partial_{[\mu} E_{\nu]}{}^{\hat{A}} + 2\varepsilon^{\hat{A}}{}_{\hat{B}\hat{C}} \Omega_{[\mu}{}^{\hat{B}} E_{\nu]}{}^{\hat{C}} - \bar{\psi}_\mu{}^i \gamma^{\hat{A}} \psi_\nu^j \delta_{ij},$$

$$R_{\mu\nu}{}^{\hat{A}}(J) \equiv 2\partial_{[\mu} \Omega_{\nu]}{}^{\hat{A}} + \varepsilon^{\hat{A}}{}_{\hat{B}\hat{C}} \Omega_\mu{}^{\hat{B}} \Omega_\nu{}^{\hat{C}},$$

$$R_{\mu\nu}(Q^i) \equiv 2\partial_{[\mu} \psi_{\nu]}^i + \Omega_{[\mu}{}^{\hat{A}} \gamma_{\hat{A}} \psi_{\nu]}^i, \tag{163}$$

equal to zero. Splitting the index $\hat{A}$ as $\hat{A} = \{0, a\}$, with $a = 1, 2$ and defining

等于零得到。将指标 $\hat{A}$ 拆分为 $\hat{A} = \{0, a\}$ ，其中 $a = 1, 2$ ，并定义

$$Q_\pm = \frac{1}{\sqrt{2}}\left(Q^1 \pm \gamma_0 Q^2\right), \ \psi_{\mu\pm} = \frac{1}{\sqrt{2}}\left(\psi_\mu^1 \pm \gamma_0 \psi_\mu^2\right), \tag{164}$$

these Maurer-Cartan equations can be rewritten as

这些 Maurer-Cartan 方程可改写为

$$R_{\mu\nu}{}^0(P) \equiv 2\partial_{[\mu} E_{\nu]}{}^0 + 2\varepsilon_{ab} \Omega_{[\mu}{}^a E_{\nu]}{}^b - \bar{\psi}_{\mu+} \gamma^0 \psi_{\nu+} - \bar{\psi}_{\mu-} \gamma^0 \psi_{\nu-} = 0,$$

$$R_{\mu\nu}{}^0(J) \equiv 2\partial_{[\mu} \Omega_{\nu]}{}^0 + \varepsilon_{ab} \Omega_{[\mu}{}^a \Omega_{\nu]}{}^b = 0$$

$$R_{\mu\nu}(Q_+) \equiv 2\partial_{[\mu} \psi_{\nu]+} + \Omega_{[\mu}{}^0 \gamma_0 \psi_{\nu]+} + \Omega_{[\mu}{}^a \gamma_a \psi_{\nu]-} = 0,$$

$$R_{\mu\nu}{}^a(P) \equiv 2\partial_{[\mu} E_{\nu]}{}^a + 2\varepsilon^a{}_b \Omega_{[\mu}{}^0 E_{\nu]}{}^b - 2\varepsilon^a{}_b \Omega_{[\mu}{}^b E_{\nu]}{}^0 - 2\bar{\psi}_{[\mu+} \gamma^a \psi_{\nu]-} = 0,$$

$$R_{\mu\nu}{}^a(J) \equiv 2\partial_{[\mu} \Omega_{\nu]}{}^a + 2\varepsilon^a{}_b \Omega_{[\mu}{}^0 \Omega_{\nu]}{}^b = 0,$$

$$R_{\mu\nu}(Q_-) \equiv 2\partial_{[\mu} \psi_{\nu]-} + \Omega_{[\mu}{}^0 \gamma_0 \psi_{\nu]-} + \Omega_{[\mu}{}^a \gamma_a \psi_{\nu]+} = 0. \tag{165}$$

Since the $\mathcal{N} = 2$ super-Poincaré algebra can be decomposed as $V_0 \oplus V_1$ with

由于 $\mathcal{N} = 2$ 超庞加莱代数可分解为 $V_0 \oplus V_1$ ，其中

$$V_0 = \left\{P^0, J^0, Q_+\right\} \ \text{ and } \ V_1 = \{P^a, J^a, Q_-\}, \tag{166}$$

obeying (156), one can perform the above outlined expansion procedure. Picking $N_0 = 2$ and $N_1 = 1$, one thus expands

满足式 (156)，因此可以执行上述概括的展开步骤。选取 $N_0 = 2$ 和 $N_1 = 1$，即可展开

$$E_\mu{}^0 = \tau_{\mu(0)} + \lambda^2 \tau_{\mu(2)},\ \Omega_\mu{}^0 = \omega_{\mu(0)} + \lambda^2 \omega_{\mu(2)},\ \psi_{\mu+} = \psi_{\mu+(0)} + \lambda^2 \psi_{\mu+(2)},$$

$$E_\mu{}^a = \lambda e_{\mu(1)}{}^a,\ \Omega_\mu{}^a = \lambda \omega_{\mu(1)}{}^a,\ \psi_{\mu-} = \lambda \psi_{\mu-(1)}. \tag{167}$$

Plugging these in the Maurer-Cartan equations (165) and setting the coefficients of like powers of λ equal to zero, one is led to the following equations:

将这些展开式代入 Maurer-Cartan 方程 (165)，并令 λ 同次幂的系数等于零，得到下述方程:

$$2\partial_{[\mu}\tau_{\nu](0)} - \overline{\psi}_{\mu+(0)}\gamma^0\psi_{\nu+(0)} = 0$$

$$2\partial_{[\mu}\tau_{\nu](2)} + 2\varepsilon^a{}_b\omega_{[\mu(1)}{}^a e_{\nu](1)}{}^b - \overline{\psi}_{\mu-(1)}\gamma^0\psi_{\nu-(1)} - 2\overline{\psi}_{[\mu+(0)}\gamma^0\psi_{\nu]+(2)} = 0,$$

$$2\partial_{[\mu}\omega_{\nu](0)} = 0$$

$$2\partial_{[\mu}\omega_{\nu](2)} + \varepsilon_{ab}\omega_{[\mu(1)}{}^a\omega_{\nu](1)}{}^b = 0$$

$$2\partial_{[\mu}\psi_{\nu]+(0)} + \omega_{[\mu(0)}\gamma_0\psi_{\nu]+(0)} = 0$$

$$2\partial_{[\mu}\psi_{\nu]+(2)} + \omega_{[\mu(0)}\gamma_0\psi_{\nu]+(2)} + \omega_{[\mu(2)}\gamma_0\psi_{\nu]+(0)} + \omega_{[\mu(1)}{}^a\gamma_a\psi_{\nu]-(1)} = 0,$$

$$2\partial_{[\mu}e_{\nu](1)}{}^a + 2\varepsilon^a{}_b\omega_{[\mu(0)}e_{\nu](1)}{}^b - 2\varepsilon^a{}_b\omega_{[\mu(1)}{}^b\tau_{\nu](0)} - 2\overline{\psi}_{\mu+(0)}\gamma^a\psi_{\nu]-(1)} = 0,$$

$$2\partial_{[\mu}\omega_{\nu](1)}{}^a + 2\varepsilon^a{}_b\omega_{[\mu(0)}\omega_{\nu](1)}{}^b = 0$$

$$2\partial_{[\mu}\psi_{\nu]-(1)} + \omega_{[\mu(0)}\gamma_0\psi_{\nu]-(1)} + \omega_{[\mu(1)}{}^a\gamma_a\psi_{\nu]+(0)} = 0. \tag{168}$$

Upon identifying

做等价替换

$$\tau_{\mu(0)} \to \tau_\mu,\ \tau_{\mu(2)} \to m_\mu,\ e_{\mu(1)}{}^a \to e_\mu{}^a,$$

$$\omega_{\mu(0)} \to \omega_\mu,\ \omega_{\mu(2)} \to s_\mu,\ \omega_{\mu(1)}{}^a \to \omega_\mu{}^a,$$

$$\psi_{\mu+(0)} \to \psi_{\mu+},\ \psi_{\mu+(2)} \to \rho_\mu,\ \psi_{\mu-(1)} \to \psi_{\mu-}, \tag{169}$$

the left-hand sides of (168) turn into the supercovariant curvatures of the gauge field (151) of the superalgebra (142), (149). This shows that (168) are the Maurer-Cartan equations of the extended Bargmann superalgebra (142), (149) and that the latter is indeed obtained as a Lie algebra expansion of the $3D\mathcal{N} = 2$ super-Poincaré algebra.

(168) 的左侧即变为超代数 (142)、(149) 中规范场 (151) 的超协变曲率。这说明 (168) 就是扩展 Bargmann 超代数 (142)、(149) 的 Maurer-Cartan 方程，后者确实是通过 $3D\mathcal{N} = 2$ 超庞加莱代数的李代数展开得到的。

Above we focused on NR Chern-Simons supergravity theories that are based on extensions of the $3D$ Bargmann superalgebra. Different types of NR Chern-Simons supergravity theories have been considered in the literature as well. For instance, theories of this kind with extensions of Lifshitz and Schrödinger symmetries were considered in [26]; a teleparallel NR supergravity theory was given (based on earlier work [56,57]) in a Chern-Simons formulation in [31]. A NR Chern-Simons supergravity theory that is based on an extension of a $c \to \infty$ limit of the Maxwell algebra (i.e., the extension of the Poincaré algebra with generators $Z_{\hat{A}\hat{B}}$ such that $\left[P_{\hat{A}}, P_{\hat{B}}\right] = Z_{\hat{A}\hat{B}}$) was constructed in [27]. For an example in two dimensions (see [32]).

上文我们聚焦于基于扩展 $3D$ Bargmann 超代数的非相对论 Chern-Simons 超引力理论。文献中也研究了不同类型的非相对论 Chern-Schrödinger 超引力理论。例如，文献 [26] 研究了这类带有 Lifshitz 对称性和薛定谔对称性扩展的理论；文献 [31] 基于前人工作 [56,57] 给出了远平行非相对论超引力理论的 Chern-Simons 表述。文献 [27] 构造了一种基于麦克斯韦代数 $c \to \infty$ 极限扩展的非相对论 Chern-Simons 超引力理论 (即带有生成元 $Z_{\hat{A}\hat{B}}$ 的庞加莱代数扩展，满足 $\left[P_{\hat{A}}, P_{\hat{B}}\right] = Z_{\hat{A}\hat{B}}$)。二维的例子可见 [32]。

This finishes our discussion of non-Lorentzian supergravity theories in $3D$. In the next section, we will focus our attention on higher dimensions and discuss recent results on $10D$ non-Lorentzian supergravity.

至此我们完成了对 $3D$ 中非洛伦兹超引力理论的讨论。下一节我们将聚焦高维情形，讨论 $10D$ 非洛伦兹超引力的最新研究结果。

10D Minimal Supergravity

10 维极小超引力

The $10D$ minimal supergravity theory is the first pure NR supergravity theory constructed in a dimension $D > 4$. It was obtained by taking the NR limit of the $10D$ relativistic $\mathcal{N} = 1$ supergravity theory without Yang-Mills matter couplings [34]. Before taking the limit, let us first define the relativistic $\mathcal{N} = 1$ supergravity theory [58, 59] . The field content is given by

$10D$ 极小超引力理论是首个在大于四维的维度 $D > 4$ 中构造出的纯非相对论超引力理论。它通过对不存在杨-米尔斯物质耦合的 $10D$ 相对论性 $\mathcal{N} = 1$ 超引力理论取非相对论极限得到 [34]。在取极限前，我们先定义相对论性 $\mathcal{N} = 1$ 超引力理论 [58, 59] 。其场内容为:

$$\left\{E_\mu{}^{\hat{A}}, B_{\mu\nu}, \Phi; \Psi_\mu, \lambda\right\}, \tag{170}$$

where $\left\{E_\mu{}^{\hat{A}}, B_{\mu\nu}, \Phi\right\}$ are the vielbein field, Kalb-Ramond field and the dilaton field, respectively, and $\{\Psi_\mu, \lambda\}$ are the left-handed Majorana-Weyl spinor gravitino field and the right-handed dilatino, respectively. For the discussion to follow, we only need the specific form of the bosonic part of the $\mathcal{N}=1$ supergravity action (The fermionic part can be found in [34] but will not be needed here.)

其中 $\left\{E_\mu{}^{\hat{A}}, B_{\mu\nu}, \Phi\right\}$ 分别是标架场、Kalb-Ramond 场和伸缩子场，$\{\Psi_\mu, \lambda\}$ 分别是左手 Majorana-Weyl 旋量引力微子场和右手伸缩微子场。在后续讨论中，我们仅需要 $\mathcal{N}=1$ 超引力作用量玻色子部分的具体形式 (费米子部分可参见文献 [34]，此处不需要)。

$$S=\frac{1}{2\kappa^2}\int \mathrm{d}^{10}x E \mathrm{e}^{-2\Phi}\left\{\mathcal{R}+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{12}\mathcal{H}_{\mu\nu\rho}\mathcal{H}^{\mu\nu\rho}\right\}, \tag{171}$$

where κ denotes the gravitational coupling constant, $E=\det\left(E_\mu{}^{\hat{A}}\right)$, $\mathcal{R}$ is the Ricci scalar and

其中 κ 表示引力耦合常数，$E=\det\left(E_\mu{}^{\hat{A}}\right)$, $\mathcal{R}$ 是里奇标量，且

$$\mathcal{H}_{\mu\nu\rho}=3\partial_{[\mu}B_{\nu\rho]} \tag{172}$$

is the field strength of the Kalb-Ramond field.

是 Kalb-Ramond 场的场强。

The fields of ten-dimensional $\mathcal{N}=1$ supergravity transform as follows under local Lorentz transformations with parameter $\Lambda^{\hat{A}\hat{B}}$, a one-form symmetry of the Kalb-Ramond field with parameter Θ_μ and supersymmetry with a left-handed Majorana-Weyl spinor parameter ε :

十维 $\mathcal{N}=1$ 超引力的场在参数为 $\Lambda^{\hat{A}\hat{B}}$ 的局部洛伦兹变换、参数为 Θ_μ 的 Kalb-Ramond 场一维变换，以及参数为左手 Majorana-Weyl 旋量 ε 的超对称变换下，变换规律如下:

$$\delta E_\mu{}^{\hat{A}}=\Lambda^{\hat{A}}{}_{\hat{B}}E_\mu{}^{\hat{B}}+\bar{\varepsilon}\Gamma^{\hat{A}}\Psi_\mu, \tag{173a}$$

$$\delta B_{\mu v}=2\partial_{[\mu}\Theta_{v]}+2\bar{\varepsilon}\Gamma_{[\mu}\Psi_{v]},\ \delta\Phi=\frac{1}{2}\bar{\varepsilon}\lambda, \tag{173b}$$

$$\delta\Psi_\mu=\frac{1}{4}\Lambda^{\hat{A}\hat{B}}\Gamma_{\hat{A}\hat{B}}\Psi_\mu+D_\mu\left(\Omega^{(+)}\right)\varepsilon, \tag{173c}$$

$$\delta\lambda=\frac{1}{4}\Lambda^{\hat{A}\hat{B}}\Gamma_{\hat{A}\hat{B}}\lambda+\Gamma^\mu\varepsilon\partial_\mu\Phi-\frac{1}{12}\Gamma^{\hat{A}\hat{B}\hat{C}}\varepsilon\mathcal{H}_{\hat{A}\hat{B}\hat{C}}, \tag{173d}$$

where the $10D$ gamma-matrices are denoted by $\Gamma_{\hat{A}}$, and we have ignored terms quadratic in the fermion fields in the supersymmetry rules of the fermions. We have furthermore defined the following torsionful covariant derivative of ε :

其中 $10D$ 伽马矩阵记作 $\Gamma_{\hat{A}}$，我们在费米子的超对称变换规则中忽略了费米场的二次项。此外我们定义了 ε 的如下含挠协变导数:

$$D_\mu\left(\Omega^{(+)}\right)\varepsilon = \partial_\mu\varepsilon - \frac{1}{4}\Omega_\mu^{(+)\hat{A}\hat{B}}\Gamma_{\hat{A}\hat{B}}\varepsilon \text{ with } \Omega_\mu^{(+)\hat{A}\hat{B}} = \Omega_\mu{}^{\hat{A}\hat{B}} + \frac{1}{2}\mathcal{H}_\mu{}^{\hat{A}\hat{B}}.$$

(174)

In order to define the NR limit, we introduce a (dimensionless) parameter ω and perform the following invertible field redefinition:

为了定义非相对论极限，我们引入一个 (无量纲) 参数 ω，并进行如下可逆场重定义:

$$\tau_\mu{}^A = \omega^{-1}E_\mu{}^A,\ e_\mu{}^a = E_\mu{}^a,\ b_{\mu\nu} = B_{\mu\nu} + \varepsilon_{AB}E_\mu{}^A E_\nu{}^B,$$

$$\phi = \Phi - \log\omega,\ \psi_{\mu\pm} = \omega^{\mp 1/2}\Pi_\pm\Psi_\mu,\ \lambda_\pm = \omega^{\mp 1/2}\Pi_\pm\lambda, \tag{175}$$

where we have split the Lorentz index $\hat{A}$ into a longitudinal index $A = 0, 1$ and a transversal index $a = 2, \ldots, 9$. Note that the redefinition of the spinor fields involves the 'worldsheet chirality' projections

其中我们将洛伦兹指标 $\hat{A}$ 分解为纵向指标 $A = 0,1$ 和横向指标 $a = 2, \ldots, 9$ 。注意旋量场的重定义包含“世界面手征”投影

$$\chi_\pm = \Pi_\pm\chi \ \text{ with } \ \Pi_\pm = \frac{1}{2}\left(\mathbb{1} \pm \Gamma_{01}\right) \ \text{ for any spinor } \chi. \tag{176}$$

In the $\omega \to \infty$ limit, the fields $\tau_\mu{}^A, e_\mu{}^a$ and $b_{\mu\nu}$ become the longitudinal and transversal vielbein and two-form field of stringy NC geometry. This can be verified by noting that the limit of the Lorentz and one-form symmetry transformation rules of the first line of (175), taken in a similar way as in the discussion around eq. (61), leads to the transformation rules (32) and (33).

在 $\omega \to \infty$ 极限下，场 $\tau_\mu{}^A, e_\mu{}^a$ 和 $b_{\mu\nu}$ 成为弦非对易几何的纵向、横向标架场和二形式场。我们可以验证这一点: 和式 (61) 附近的讨论采用类似方式取极限后，式 (175) 第一行的洛伦兹和一维对称变换规则会得出变换规则 (32) 和 (33)。

The following issues complicate taking the $\omega \to \infty$ limit of the action and supersymmetry transformation rules of $\mathcal{N} = 1$ supergravity:

以下问题给 $\omega \to \infty$ 超引力作用量和超对称变换规则取 $\mathcal{N} = 1$ 极限带来了复杂性:

1. The emergence of both bosonic and fermionic Stueckelberg symmetries

1. 玻色子和费米子施蒂克尔贝格对称性都会出现

2. The occurrence of divergent terms in the action

2. 作用量中会出现发散项

3. The presence of divergent terms in the supersymmetry transformation rules

3. 超对称变换规则中存在发散项

On top of this, after taking the limit, the two-form field $b_{\mu\upsilon}$, in contrast to its relativistic counterpart $B_{\mu\nu}$, transforms under Galilean boosts (see Eq. (32)) and should therefore be considered on par with the vielbein fields ${\tau_\mu}^A$ and ${e_\mu}^a$. The above three complications lead to the following new features [34].

除此之外，取极限后，二形式场 $b_{\mu\upsilon}$ 与它的相对论对应场 $B_{\mu\nu}$ 不同，它会在伽利略 boost 下变换 (参见式 (32))，因此应该将它与标架场 ${\tau_\mu}^A$ 和 ${e_\mu}^a$ 同等看待。上述三个复杂性引出了如下新特征 [34]。

1. Due to the emergent Stueckelberg symmetries, the NR superalgebra is realized on a smaller set of field components than in the relativistic case.

1. 由于施蒂克尔贝格对称性的出现，非相对论超代数是在比相对论情况更少的一组场分量上实现的。

2. The limit we take is critical in the sense that it is defined in such a way that the divergent terms in the action cancel amongst each other.

2. 我们所取的极限是临界的，因为它的定义方式使得作用量中的发散项彼此抵消。

3. The divergent terms in the supersymmetry rules are tamed by imposing geometric constraints that we will identify below.

3. 超对称规则中的发散项可通过施加我们下文将确定的几何约束得到控制。

Restricting to the bosonic case, the emergence of local dilatation Stueckelberg symmetries can already be seen by looking at the symmetries of the NR Polyakov action [45, 60] . These scaling symmetries act non-isotropically in the sense that the vielbein fields transform as follows:

限于玻色子情形，通过考察非相对论 Polyakov 作用量 [45, 60] 的对称性，已经可以看出局部伸缩斯图克尔伯格对称性的出现。这些标度对称性是非各同性作用的，具体而言，标架场的变换如下:

$$\delta_D {\tau_\mu}^A = \lambda_D {\tau_\mu}^A,\ \delta {e_\mu}^a = 0. \tag{177}$$

The fact that the transversal vielbein are scale-invariant is typical for the string. A similar anisotropic scaling symmetry occurs in the $11D$ membrane case where the transversal vielbein have a nonzero scaling weight [61]. Besides the vielbein the only other bosonic field that is not scale-invariant is the dilaton:

横向标架具有标度不变性是弦的典型特征。类似的各向异性标度对称性也出现在 $11D$ 膜情形中，此时横向标架具有非零标度权重 [61]。除标架外，唯一不具有标度不变性的其他玻色场就是 dilation:

$$\delta\phi = \lambda_D. \tag{178}$$

From this, it follows that the combination $e^{-\phi}{\tau_\mu}^A$ is scale-invariant. Usually, the (vacuum expectation value of) the dilaton is associated with the string coupling constant g_s and, when the longitudinal spatial direction is compactified (as is the case in NR string theory), ${\tau_\mu}^1$ encodes the radius R of the compactification circle. The above scaling symmetries imply that only a product of the dilaton and ${\tau_\mu}^1$ has an invariant meaning. This means that effectively we only have one invariant modulus instead of two.

由此可得，组合 $e^{-\phi}\tau_\mu{}^A$ 是标度不变的。通常，dilaton 的 (真空期望值) 与弦耦合常数 g_s 相关联，并且当纵向空间方向被紧致化时 (非相对论弦理论即是如此)，$\tau_\mu{}^1$ 编码了紧致化圆周的半径 R 。上述标度对称性意味着，只有 dilaton 与 $\tau_\mu{}^1$ 的乘积具有不变意义。这意味着实际上我们仅拥有一个不变模，而非两个。

In the supersymmetric case, there are additional fermionic fields that transform under the scaling symmetries. Since we lack a NR supersymmetric Green-Schwarz superstring sigma model, it is easiest to derive the scaling weights of the fermionic fields by first deriving the fermionic Stueckelberg symmetries. To derive the latter, we can make use of the fact that there are divergent terms in the action that cancel against each other. To discuss these cancellations, we consider the redefinitions (175) that define the NR limit. Substituting these redefinitions into the relativistic supergravity action leads to an expansion in powers of ω^{-2} of the form

在超对称情形下，存在额外的费米子场，它们会在标度对称性下变换。由于目前还没有非相对论超对称格林-施瓦茨超弦 sigma 模型，通过先推导费米子斯图克尔伯格对称性来得到费米子场的标度权重是最简便的方法。为了推导后者，我们可以利用作用量中存在相互抵消的发散项这一事实。为了讨论这些抵消，我们考虑定义非相对论极限的重定义 (175)。将这些重定义代入相对论超引力作用量，会得到按 ω^{-2} 幂次展开的如下形式

$$S = \omega^2 S^{(2)} + S^{(0)} + \omega^{-2} S^{(-2)} + \omega^{-4} S^{(-4)}, \tag{179}$$

where each of the $S^{(i)}$ now depends on the fields $\tau_\mu{}^A, e_\mu{}^a, b_{\mu\nu}, \phi, \psi_{\mu\pm}$ and $\lambda_\pm$. The fact that we find that

其中每一个 $S^{(i)}$ 现在都依赖于场 $\tau_\mu{}^A, e_\mu{}^a, b_{\mu\nu}, \phi, \psi_{\mu\pm}$ 和 $\lambda_\pm$ 。我们发现

$$S^{(2)} = 0 \tag{180}$$

combined with the fact that the action is supersymmetric leads to the emergence of fermionic Stueckelberg symmetries as follows. Due to the occurrence of divergent terms in the supersymmetry rules, we can write the infinitesimal action δ_Q of a generic supersymmetry Q as follows:

结合作用量是超对称的这一事实，费米子斯图克尔伯格对称性的出现过程如下。由于超对称规则中存在发散项，我们可以将任意超对称 Q 的无穷小作用 δ_Q 写作如下形式:

$$\delta_Q F = \omega^2 \delta_Q^{(2)} F + \delta_Q^{(0)} F + \omega^{-2} \delta_Q^{(-2)} F, \tag{181}$$

where F is any of the fields $\tau_\mu{}^A, e_\mu{}^a, b_{\mu\nu}, \phi, \psi_{\mu\pm}, \lambda_\pm$. The supersymmetry variation $\delta_Q S$ of the action can then be expanded as

其中 F 是任意场 $\tau_\mu{}^A, e_\mu{}^a, b_{\mu\nu}, \phi, \psi_{\mu\pm}, \lambda_\pm$ 。作用量的超对称变分 $\delta_Q S$ 于是可以展开为

$$\delta_Q S = \omega^2 \delta_Q^{(2)} S^{(0)} + \omega^0 \left(\delta_Q^{(0)} S^{(0)} + \delta_Q^{(2)} S^{(-2)} \right) + \mathcal{O}\left(\omega^{-2}\right). \tag{182}$$

Requiring invariance of S imposes that every order of ω^{-2} in (182) is separately zero. This in particular leads to the following two requirements:

要求 S 具有不变性，意味着式 (182) 中 ω^{-2} 的每一阶都必须分别为零。这尤其会给出以下两个条件:

(1)

$$(1)\ \delta_Q^{(2)} S^{(0)} = 0\,, \tag{183}$$

$$(2)\ \delta_Q^{(0)} S^{(0)} = -\delta_Q^{(2)} S^{(-2)}\,. \tag{184}$$

From the first condition it follows that, when varying $S^{(0)}$, all divergent terms in the supersymmetry rules must cancel amongst each other. It turns out that these divergent terms only occur in the supersymmetry variation of two fermionic field components and that they occur in the form of a shift transformation. From this, it follows that the cancellation of the divergences is equivalent to the statement that the action $S^{(0)}$ is invariant under a set of fermionic Stueckelberg symmetries. A more detailed analysis shows that these Stueckelberg symmetries are given by [34] (Here and in the following, we will often use a light-cone notation $\pm$ for the index $A : A = (+,-)$, defined according to e.g., $\tau_\mu{}^{\pm} = \left(1/\sqrt{2}\right)\left(\tau_\mu{}^0 \pm \tau_\mu{}^1\right)$.)

从第一个条件可得，当对 $S^{(0)}$ 变分时，超对称规则中的所有发散项必须相互抵消。可以发现这些发散项仅出现在两个费米子场分量的超对称变换中，且以平移变换的形式存在。由此可得，发散项抵消等价于作用量 $S^{(0)}$ 在一组费米型施蒂克尔贝格对称性下不变。更详细的分析表明这些施蒂克尔贝格对称性由文献 [34] 给出 (下文中，我们经常对指标 $A : A = (+,-)$ 使用光锥记号 $\pm$ ，其定义例如可参见 $\tau_\mu{}^{\pm} = \left(1/\sqrt{2}\right)\left(\tau_\mu{}^0 \pm \tau_\mu{}^1\right)$ 。)

$$\delta_S \psi_{\mu+} = \frac{1}{2}\tau_\mu{}^+ \Gamma_+ \eta_-,\ \delta_S \lambda_- = \eta_-,$$

$$\delta_T \psi_{\mu-} = \tau_\mu{}^+ \rho_-, \tag{185}$$

where η_- and ρ_- are the parameters of the shift symmetries that we will refer to as S - and T -supersymmetries in what follows.

其中 η_- 和 ρ_- 是平移对称性的参数，下文我们将其分别称为 S 超对称性和 T 超对称性。

Whereas the divergent parts $\delta_Q^{(2)}$ of the supersymmetry transformation rules can, according to the above discussion, be identified as (special cases of) the fermionic S - and T -Stueckelberg symmetries, the $\delta_Q^{(0)}$ part has the right structure to give NR supersymmetry transformation rules. Note however that the second condition given in Eq. (184) then implies that $S^{(0)}$, the NR limit of the action, is not invariant under NR supersymmetry. This can be remedied by imposing by hand additional geometric constraints setting certain so-called geometric tensors equal to zero. The occurrence of these geometric tensors is characteristic of NC and stringy NC geometry. In the current context, they are defined by dividing the properly supercovariantized curvature tensors $T_{\mu\nu}{}^A$ and $T_{\mu\nu}{}^a$ of the vielbein $\tau_\mu{}^A$ and $e_\mu{}^a$ as well as the supercovariantized curvature tensor $h_{\mu\nu\rho}$ of the two-form $b_{\mu\nu}$ into conventional tensors and geometric tensors as follows (Note that a similar division does not apply to the supercovariantized curvatures of the spin connection fields.):

根据上述讨论，超对称变换规则的发散部分 $\delta_O^{(2)}$ 可以被识别为费米型 S 和 T 施蒂克尔贝格对称性 (特例)，而 $\delta_O^{(0)}$ 部分具有合适的结构，可以给出非相对论超对称变换规则。但需注意，式 (184) 给出的第二个条件意味着作用量的非相对论极限 $S^{(0)}$ 在非相对论超对称性下并不不变。这一问题可以通过手动添加额外几何约束解决: 该约束将特定的所谓几何张量置零。这类几何张量的出现是非对易 (NC) 几何与弦论非对易几何的特征。在当前语境下，对它们的定义是: 将标架 $\tau_\mu{}^A$ 、$e_\mu{}^a$ 经过适当超协变化得到的曲率张量 $T_{\mu\nu}{}^A$ 、$T_{\mu\nu}{}^a$ ，以及二形式 $b_{\mu\nu}$ 经过超协变化得到的曲率张量 $h_{\mu\nu\rho}$ 按照如下方式拆分为常规张量与几何张量 (注意: 类似拆分不适用于自旋联络场的超协变曲率。):

A geometric tensor is a curvature component that does not contain (components of) a stringy NC spin connection (for longitudinal Lorentz transformations, transversal spatial rotations or string Galilean boosts) or the dilatation gauge field. Setting a geometric tensor to zero leads to a geometric constraint on the underlying geometry. A conventional tensor is a curvature component that contains a stringy NC spin connection (for longitudinal Lorentz transformations, transversal spatial rotations or string Galilean boosts) and/or the dilatation gauge field, multiplied by an invertible longitudinal or transverse Vierbein field. Setting a conventional tensor to zero leads to a conventional constraint that can be used to solve for a spin connection or dilatation gauge field component.

几何张量是不包含弦论非对易自旋联络 (对应纵向洛伦兹变换、横向空间转动或弦伽利略 boost) 分量或伸缩规范场的曲率分量。将几何张量置零会对基础几何施加几何约束。常规张量是包含弦论非对易自旋联络 (对应纵向洛伦兹变换、横向空间转动或弦伽利略 boost) 和/或伸缩规范场、再乘以可逆纵向或横向标架场的曲率分量。将常规张量置零得到的常规约束可用于求解自旋联络或伸缩规范场分量。

Using these definitions, we find the following geometric tensors (Note the similarity with the discussion around Eq. (38) in section "Stringy Newton-Cartan

利用这些定义，我们得到如下几何张量 (注意这与“弦牛顿-卡坦

Geometry." The main difference between section "Stringy Newton-Cartan Geometry" and the current discussion is that the former does not include fermionic symmetries and the dilatation symmetry. In particular, the absence of dilatations is responsible for the fact that in section "Stringy Newton-Cartan Geometry," $T_{a(AB)}$ appears as a geometric tensor, instead of $T_{a\{AB\}}$ here.)

几何”一节中式 (38) 附近的讨论类似。“弦牛顿-卡坦几何”一节与当前讨论的主要区别是，前者不包含费米对称性与伸缩对称性。具体而言，正是因为没有伸缩对称性，在“弦牛顿-卡坦几何”一节中，$T_{a(AB)}$ 才作为几何张量出现，此处则是 $T_{a\{AB\}}$ 。)

$$T_{ab}{}^A, T_a{}^{\{AB\}}, h_{abc}, \tag{186}$$

where $\{AB\}$ indicates the symmetric traceless part of AB . All other components are conventional tensors, and setting them to zero can be used to solve for the spin connection fields and dilatation gauge field except for the components

其中 $\{AB\}$ 表示 AB 的对称无迹部分。所有其余分量都是常规张量，将它们置零可用于求解自旋联络场和伸缩规范场，下述分量除外:

$$\omega_{\{AB\}a} \text{ and } b_A\,. \tag{187}$$

It turns out that, in order to ensure that $S^{(0)}$ is invariant under NR supersymmetry, we need to set to zero the following subset of geometric tensors:

事实证明，为了确保 $S^{(0)}$ 在非相对论超对称下不变，我们需要将以下几何张量子集置零:

$$T_{ab}{}^{-} = T_{a+}{}^{-} = 0. \tag{188}$$

These geometric constraints can equivalently be described by the following foliation constraint:

这些几何约束可以等价地由如下叶状结构约束描述:

$$\tau_{[\mu}{}^{-}\partial_\nu \tau_{\rho]}{}^{-} = 0. \tag{189}$$

Once we have established that the action $S^{(0)}$, after taking the limit, is invariant under the regular NR Q -supersymmetry and the new emergent S - and T - supersymmetries, we are also able to derive the emergent anisotropic scale or D -symmetry. The easiest way to derive the D -symmetry is to require that the action $S^{(0)}$ must be invariant under the commutator of a Q - with a S - or T - supersymmetry, i.e.,:

当我们确定取极限后的作用量 $S^{(0)}$ 在常规非相对论 Q 超对称，以及新涌现的 S 和 T 超对称下不变后，我们还可以推导出涌现的各向异性标度对称性即 D 对称性。推导 D 对称性最简单的方法是要求作用量 $S^{(0)}$ 在 Q 超对称与 S 或 T 超对称的对易子下不变，即:

$$\left[\bar{\varepsilon}Q, \bar{\eta}S \text{ or } \bar{\eta}T\right] \sim \lambda_D D. \tag{190}$$

This yields precisely the anisotropic scale transformations given in Eqs. (177) and (178) together with

由此恰好得到式 (177) 和 (178) 给出的各向异性标度变换，再加上

$$\delta_D \psi_{\mu\pm} = \pm\frac{1}{2}\lambda_D \psi_{\mu\pm},\ \delta_D \lambda_\pm = \pm\frac{1}{2}\lambda_D \lambda_\pm. \tag{191}$$

An important simplifying feature of the $\mathcal{N} = 1$ supersymmetric case is that the geometric constraints (188) are invariant under NR supersymmetry and therefore do not lead to further constraints. The constraints should be imposed with care: they should not be substituted in the action $S^{(0)}$ but only in the supersymmetry variation of the action. Actually, the action $S^{(0)}$ serves the purpose of a pseudo-action: it is a convenient way to derive a subset of the equations of motion that we denominate the 'bulk' equations of motion B . Due to the emerging anisotropic dilatations and Stueckelberg symmetries, there are also so-called 'missing' equations of motion M that can only be derived by taking the NR limit of the equations of motion instead of the action. The complete set of equations of motion $\{M, B\}$ forms a reducible indecomposable representation in the sense that under Galilean boosts G we have

$\mathcal{N}=1$ 超对称情形的一个重要简化特征是，几何约束 (188) 在非相对论超对称下不变，因此不会带来额外约束。施加这些约束时需要注意: 不应将它们代入作用量 $S^{(0)}$，仅需代入作用量的超对称变分。实际上，作用量 $S^{(0)}$ 起到伪作用量的作用: 它是推导一部分运动方程的便捷方法，我们将这部分运动方程称为「体」运动方程 B。由于涌现出的各向异性膨胀和施蒂克尔贝格对称性，还存在所谓的「缺失」运动方程 M，这类方程只能通过对运动方程而非作用量取非相对论极限得到。完整的运动方程集合 $\{M, B\}$ 构成一个可约不可分解表示，在伽利略 boost G 下我们有

$$\delta_G M \sim B \ \text{ but not } \ \delta_G B \sim M. \tag{192}$$

This is not the case for the supersymmetry transformations Q that connects all equations of motion back and forth, i.e.,:

这一点对超对称变换 Q 不成立，超对称变换可以将所有运动方程相互关联，即:

$$\delta_Q M \sim B \ \text{ and } \ \delta_Q B \sim M. \tag{193}$$

The reason that the bulk equations of motion do not form a separate multiplet is that the action $S^{(0)}$ is only invariant under supersymmetry after imposing the geometric constraints in the supersymmetry variation of the action. Such an action does not satisfy the criteria considered in [62].

体运动方程不能单独构成一个多重态的原因是，只有在作用量的超对称变分中施加几何约束后，作用量 $S^{(0)}$ 才具有超对称不变性。这类作用量不满足文献 [62] 中给出的判据。

The above shows that taking the non-Lorentzian limit of a relativistic action and next determining the equations of motion of the resulting non-Lorentzian action is not the same as varying the relativistic action and then taking the non-Lorentzian limit of the resulting relativistic equations of motion. Nevertheless, the construction of a non-Lorentzian pseudo-action is a useful tool to collect all bulk equations of motion with just one action.

上述结果表明，对相对论作用量取非洛伦兹极限，再推导得到的非洛伦兹作用量的运动方程，不同于先对相对论作用量变分得到运动方程，再对这些相对论运动方程取非洛伦兹极限。尽管如此，构造非洛伦兹伪作用量仍然是收集所有体运动方程的有用工具，仅用一个作用量即可完成。

At the end of the day, the final result for $10D$ minimal supergravity is a set of constraint equations (originally called bulk equations of motion, missing equations of motion and geometric constraints) that forms a closed collection under all the symmetries of the model. In the absence of a true action, it is irrelevant to distinguish between equations of motion and additional constraint equations. More details about minimal supergravity can be found in [34]. For later reference, we give the full final answer, in a self-explanatory way, in a separate subsection below.

最终，$10D$ 极小超引力的结果是一组约束方程 (最初分为体运动方程、缺失运动方程和几何约束)，这组方程在模型的所有对称性下构成闭合集合。在不存在真实作用量的情况下，区分运动方程和额外约束方程并无意义。关于极小超引力的更多细节可以参见文献 [34]。为方便后续参考，我们会在下文中单独以一个小节给出完整的最终结果，表述自成体系。

The Complete Result

完整结果

The purpose of this subsection is to present, up to quartic fermion terms, the relevant expressions of $10D$NR minimal supergravity, including some basic definitions, in a self-contained manner that can be used for later reference. It is useful to split the NR action S_{NR} into a part S_B that is purely bosonic, a part $S_{\psi\psi}$ that is quadratic in the gravitini $\psi_{\mu\pm}$, a part $S_{\lambda\lambda}$ that is quadratic in the dilatini $\lambda_{\pm}$ and a remaining quadratic fermion part $S_{\lambda\psi}$ that contains both a gravitino and a dilatino:

本小节的目的是自洽地给出直至四次费米子项的 $10D$NR 最小超引力的相关表达式，包括一些基本定义，以便后续参考。将非相对论作用量 S_{NR} 拆分为纯玻色子部分 S_B 、引力微子 $\psi_{\mu\pm}$ 的二次项部分 $S_{\psi\psi}$ 、伸缩微子 $\lambda_{\pm}$ 的二次项部分 $S_{\lambda\lambda}$ 以及包含引力微子和伸缩微子的剩余二次费米子部分 $S_{\lambda\psi}$ 是很有用的:

$$S_{NR}=S_B+S_{\lambda\lambda}+S_{\lambda\psi}+S_{\psi\psi}+\text{ quartic fermion terms.} \tag{194}$$

As mentioned above, we will ignore all quartic fermion terms and only require supersymmetry up to cubic fermion terms.

如上所述，我们将忽略所有四次费米子项，仅要求超对称对三次费米子项成立。

The bosonic part S_B of the action has been given in [60] and reads:

作用量的玻色子部分 S_B 已在文献 [60] 中给出，形式如下:

$$\begin{aligned}S_B=\frac{1}{2\kappa^2}\int \mathrm{d}^{10}x e e^{-2\phi}\Big(&\mathrm{R}(J)+4\partial_a\phi\partial^a\phi-\frac{1}{12}h_{abc}h^{abc}\\ &-4e_a{}^{\mu}\left(\partial_\mu b^a-\omega_\mu{}^{ab}b_b-\omega_\mu{}^{Ab}\tau^a{}_{bA}\right)\\ &-4b_ab^a-4\tau_{a\{AB\}}\tau^{a\{AB\}}\Big),\end{aligned} \tag{195a}$$

where $e=\det\left(\tau_\mu{}^A,e_\mu{}^a\right),h_{\mu\upsilon\rho}=3\partial_{[\mu}b_{\upsilon\rho]}$ and $\mathrm{R}(J)=-e_a{}^{\mu}e_b{}^{\upsilon}R_{\mu\upsilon}(J)^{ab}$ with

其中 $e=\det\left(\tau_\mu{}^A,e_\mu{}^a\right),h_{\mu\upsilon\rho}=3\partial_{[\mu}b_{\upsilon\rho]}$ 和 $\mathrm{R}(J)=-e_a{}^{\mu}e_b{}^{\upsilon}R_{\mu\upsilon}(J)^{ab}$ 满足

$$\begin{aligned}\mathrm{R}_{\mu\upsilon}(J)^{ab}=&\,2\partial_{[\mu}\omega_{\upsilon]}{}^{ab}+2\omega_{[\mu}{}^{ac}\omega_{\upsilon]}{}^{b}{}_{c}\\ &+2e_{[\mu}{}^{c}\Big(2\omega_{\upsilon]}{}^{C[a}\tau^{b]}{}_{cC}-\omega_{\upsilon]Cc}\tau^{abC}\Big)^{ac}\\ &+8\tau_{[\mu}{}^{A}\left(\omega_{\upsilon]}{}^{B[a}\tau^{b]}{}_{\{AB\}}-\frac{1}{8}\varepsilon_A{}^{B}\omega_{\upsilon]Bc}h^{abc}\right).\end{aligned} \tag{195b}$$

The different dependent gauge fields occurring in the above expressions are given by

上述表达式中出现的各个依赖规范场定义如下

$$b_\mu = e_\mu{}^a \tau_{aA}{}^A + \tau_\mu{}^A \partial_A \phi, \tag{195ca}$$

$$\omega_\mu = \left(\tau_\mu{}^{AB} - \frac{1}{2}\tau_\mu{}^C \tau^{AB}{}_C\right)\varepsilon_{AB} - \tau_\mu{}^A \varepsilon_{AB}\partial^B\phi, \tag{195cb}$$

$$\omega_\mu{}^{Aa} = -e_\mu{}^{Aa} + e_{\mu b}e^{Aab} + \frac{1}{2}\varepsilon^A{}_B h_\mu{}^{Ba} + \tau_{\mu B}W^{BAa}, \tag{195cc}$$

$$\omega_\mu{}^{ab} = -2e_\mu{}^{[ab]} + e_{\mu c}e^{abc} - \frac{1}{2}\tau_\mu{}^A \varepsilon_{AB} h^{Bab}, \tag{195cd}$$

where

其中

$$\tau_{\mu\nu}{}^A = \partial_{[\mu}\tau_{\nu]}{}^A \quad \text{and} \quad e_{\mu\nu}{}^a = \partial_{[\mu}e_{\nu]}{}^a. \tag{195d}$$

Note that not all components of the above spin connections can be solved for, which is reflected by the undetermined W^{ABa} which is traceless symmetric in the (AB) indices, but otherwise arbitrary. Since all the relevant expressions - such as action, equations of motion and symmetry transformation rules - follow from a limit, it is clear that nothing depends on W^{ABa} .

注意，上述自旋联络并非所有分量都可求解，这一点体现为存在未确定的 W^{ABa} : 它在 (AB) 指标下是无迹对称的，除此之外任意。由于所有相关表达式——如作用量、运动方程和对称变换规则——都来自一个极限，因此显然没有任何物理量依赖于 W^{ABa} 。

The part of the action that is quadratic in the dilatini reads

作用量中伸缩微子二次项部分形式如下

$$S_{\lambda\lambda} = \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x e \mathrm{e}^{-2\phi}\Big(2\bar{\lambda}_\pm \Gamma^a D_a \lambda_\mp + 2\bar{\lambda}_+ \Gamma^A D_A \lambda_+$$

$$-\frac{1}{6}h_{abc}\left(\bar{\lambda}_+ \Gamma^{abc}\lambda_-\right) + \tau_{bcA}\left(\bar{\lambda}_- \Gamma^{bcA}\lambda_-\right)\Big),$$

(195e)where the covariant derivatives are covariant with respect to the Galilean symmetries and dilatations. The notation $\bar{\lambda}_\pm \Gamma \lambda_\mp$ is a shorthand for $\bar{\lambda}_+ \Gamma \lambda_- + \bar{\lambda}_- \Gamma \lambda_+$ and will be used also below.

其中协变导数是关于伽利略对称性和伸缩变换的协变导数。记号 $\bar{\lambda}_\pm \Gamma \lambda_\mp$ 是 $\bar{\lambda}_+ \Gamma \lambda_- + \bar{\lambda}_- \Gamma \lambda_+$ 的简写，下文也将沿用该约定。

Next, the off-diagonal terms in the action read

接下来，作用量中的非对角项形式如下

$$S_{\lambda\psi} = \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x e e^{-2\phi}\Big(-4\overline{\lambda}_{\pm}\Gamma^{ab}e_a{}^{\mu}e_b{}^{\upsilon}D_{[\mu}\psi_{\upsilon]\mp} - 8\overline{\lambda}_{+}\Gamma^{Ab}\tau_A{}^{\mu}e_b{}^{\upsilon}D_{[\mu}\psi_{\upsilon]+}$$

$$-4\overline{\lambda}_{\pm}\Gamma^{ab}\psi_{a\mp}D_b\phi - 4\overline{\lambda}_{+}\Gamma^{Ab}\psi_{A+}D_b\phi$$

$$+\frac{1}{6}h_{abc}\left(\overline{\lambda}_{\pm}\Gamma^{abcd}\psi_{d\mp}\right) + \frac{1}{2}h_{abc}\left(\overline{\lambda}_{+}\Gamma^{abcD}\psi_{D+}\right)$$

$$-\left(\eta^{DA} + \varepsilon^{DA}\right)\tau_{bcD}\left(\overline{\lambda}_{-}\Gamma^{bc}\psi_{A+} - \overline{\lambda}_{+}\Gamma^{bc}\psi_{A-}\right)$$

$$+2\tau_{bc}{}^{A}\overline{\lambda}_{\pm}\Gamma^{bc}\psi_{A\mp} + 2\tau^{c\{AB\}}\overline{\lambda}_{+}\Gamma_{cA}\psi_{B+}$$

$$-2\tau_{bcA}\overline{\lambda}_{-}\Gamma^{Abcd}\psi_{d-}\Big)\ . \tag{195f}$$

Finally, the pure gravitino terms are given by

最后，纯引力微子项由下式给出

$$S_{\psi\psi} = \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x e e^{-2\phi} \times \Big(-2\overline{\psi}_{A+}\Gamma^{Abc}e_b{}^{\mu}e_c{}^{\upsilon}D_{[\mu}\psi_{\upsilon]+}$$

$$-4\overline{\psi}_{a+}\Gamma^{abC}e_b{}^{\mu}\tau_C{}^{\upsilon}D_{[\mu}\psi_{\upsilon]+}$$

$$-2\overline{\psi}_{a\pm}\Gamma^{abc}e_b{}^{\mu}e_c{}^{\upsilon}D_{[\mu}\psi_{\upsilon]\mp} + \frac{1}{2}h^{abc}\left(\overline{\psi}_{a\pm}\Gamma_b\psi_{c\mp}\right)$$

$$-\frac{1}{6}h_{abc}\left(\overline{\psi}_{d+}\Gamma^{abcdE}\psi_{E+} + \frac{1}{2}\overline{\psi}_{d\pm}\Gamma^{abcde}\psi_{e\mp}\right) +$$

$$-4\left(\overline{\psi}_{a\pm}\Gamma^{a}\psi_{b\pm} + \overline{\psi}_{A+}\Gamma^{A}\psi_{b+}\right)D^b\phi$$

$$-2\left(\eta^{AD} + \varepsilon^{AD}\right)\tau^{bc}{}_{D}\overline{\psi}_{c\pm}\Gamma_b\psi_{A\mp} + 2\tau^{bcA}\left(\overline{\psi}_{b-}\Gamma_A\psi_{C'-}\right)$$

$$-2\left(\eta_{BC} - \varepsilon_{BC}\right)\tau^{c\{AB\}}\overline{\psi}^{C} + \Gamma_A\psi_{c+}$$

$$+\left(\eta_{AB} + \varepsilon_{AB}\right)\tau_{bc}{}^{A}\overline{\psi}_{d\pm}\Gamma^{BbcdE}\psi_{E\mp}$$

$$+\tau_{bc}{}^{A}\overline{\psi}_{d-}\Gamma_A\Gamma^{bcde}\psi_{e-}\Big). \tag{195g}$$

The NR supersymmetry transformation rules that leave the action S_{NR} defined above invariant (up to cubic fermion terms), upon imposition of the geometric constraints (188), are given by

在施加几何约束 (188) 后，使上述定义的作用量 S_{NR} 保持不变 (在三次费米子项阶不变) 的非相对论超对称变换规则如下

$$\delta\tau_\mu{}^A = \bar{\varepsilon}_+\Gamma^A\psi_{\mu+}, \tag{4a}$$

$$\delta e_\mu{}^a = \bar{\varepsilon}_+\Gamma^a\psi_{\mu-} + \bar{\varepsilon}_-\Gamma^a\psi_{\mu+}, \tag{4b}$$

$$\delta\phi = \frac{1}{2}\left(\bar{\varepsilon}_+\lambda_- + \bar{\varepsilon}_-\lambda_+\right), \tag{4c}$$

$$\delta b_{\mu\upsilon} = 4\tau_{[\mu}{}^A\bar{\varepsilon}_-\Gamma_A\psi_{\upsilon]-} + 2\left(e_{[\mu}{}^a\bar{\varepsilon}_+\Gamma_a\psi_{\upsilon]-} + e_{[\mu}{}^a\bar{\varepsilon}_-\Gamma_a\psi_{\upsilon]+}\right). \tag{4d}$$

as far as the bosonic fields are concerned.

以上是玻色场的变换规则。

Decomposing the supersymmetry rules of the gravitino and dilatino as follows:

我们将引力微子和伸缩微子的超对称规则做如下分解:

$$\delta\psi_{\mu+} = \delta_+\psi_{\mu+} + \delta_-\psi_{\mu-}, \tag{5a}$$

$$\delta\psi_{\mu-} = \delta_+\psi_{\mu-} + \delta_-\psi_{\mu-}, \tag{5b}$$

$$\delta\lambda_+ = \delta_+\lambda_+ + \delta_-\lambda_+, \tag{5c}$$

$$\delta\lambda_- = \delta_+\lambda_- + \delta_-\lambda_- \tag{5d}$$

we find that the supersymmetry rules of the fermionic fields are, up to terms quadratic in $\psi_{\mu\pm}, \lambda_\pm$, given by

我们得到，费米场的超对称规则在 $\psi_{\mu\pm}, \lambda_\pm$ 二次项阶以下由下式给出

$$\delta_+\psi_{\mu+} = \mathcal{D}_\mu\varepsilon_+ - \frac{1}{8}e_{\mu c}h^{cab}\Gamma_{ab}\varepsilon_+, \tag{6a}$$

$$\delta_-\psi_{\mu+} = \left(e_{\mu b}\tau^{ba+} + \tau_\mu{}^-\tau^{a++}\right)\Gamma_{a+}\varepsilon_-, \tag{6b}$$

$$\delta_+\psi_{\mu-} = -\frac{1}{2}\omega_\mu{}^{-a}\Gamma_{-a}\varepsilon_+, \tag{6c}$$

$$\delta_-\psi_{\mu-} = \mathcal{D}_\mu\varepsilon_- - \frac{1}{8}e_{\mu c}h^{cab}\Gamma_{ab}\varepsilon_-, \tag{6d}$$

$$\delta_+\lambda_+ = \left(\partial_a\phi\Gamma^a - b_a\Gamma^a - \frac{1}{12}h^{abc}\Gamma_{abc}\right)\varepsilon_+, \tag{6e}$$

$$\delta_-\lambda_+ = \frac{1}{2}\tau^{ab+}\Gamma_{ab+}\varepsilon_-, \tag{6f}$$

$$\delta_+\lambda_- = 0 \tag{6g}$$

$$\delta_-\lambda_- = \left(\partial_a\phi\Gamma^a - b_a\Gamma^a - \frac{1}{12}h^{abc}\Gamma_{abc}\right)\varepsilon_-, \tag{6h}$$

where the covariant derivative $\mathcal{D}_\mu\varepsilon_\pm$ is given by

其中协变导数 $\mathcal{D}_\mu\varepsilon_\pm$ 由下式给出

$$\mathcal{D}_\mu\varepsilon_\pm = \left(\partial_\mu - \frac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab} \pm \frac{1}{2}\omega_\mu \mp \frac{1}{2}b_\mu\right)\varepsilon_\pm. \tag{7}$$

This finishes our presentation of the action and symmetries of the 10D minimal supergravity theory.

至此我们完成了对 10D 极小超引力理论的作用量与对称性的介绍。

Conclusions

结论

In this review, we gave an overview of the different non-Lorentzian supergravity theories that have been constructed so far in the literature. In 3D and 10D , we explained the construction method based on taking a non-Lorentzian limit of a relativistic supergravity theory. Moreover, we gave the explicit results for the non-Lorentzian supergravity theory in these two cases. We also illustrated the construction of several 3D Chern-Simons supergravity theories using the Lie algebra and/or semigroup expansion.

在这篇综述中，我们概述了目前文献中已构造出的各类非洛伦兹超引力理论。在 3D 和 10D 中，我们阐释了基于相对论超引力理论取非洛伦兹极限的构造方法，此外我们给出了这两种情况下非洛伦兹超引力理论的显式结果。我们还说明了如何利用李代数和/或半群展开构造多个 3D 陈-西蒙斯超引力理论。

Obviously, more work needs to be done. At the time of writing this review, efforts are made to extend the work of [61] and to take the non-Lorentzian limit of 11D supergravity based upon a non-Lorentzian geometry with a membrane distribution of co-dimension 3 [63]. It is expected that the gauge fixing of this theory leads to a 11D supersymmetric version of Newtonian gravity much in the same spirit of the 3D Newtonian supergravity theory we discussed in this work. We expect that in the same way non-Lorentzian versions of 10D IIA and IIB supergravity can be constructed. Another issue that needs attention is a heterotic extension of the 10D minimal supergravity theory that we discussed in this review. This theory should contain a non-Lorentzian version of the Yang-Mills and Lorentz Chern-Simons term that has played such an important role in the Green-Schwarz anomaly cancellation mechanism [64]. Such anomaly cancellations are expected to also happen in the non-Lorenzian case.

显然，仍有更多工作有待完成。撰写这篇综述时，已有研究团队正在推进扩展文献 [61] 的工作，对基于余维 3 膜分布非洛伦兹几何的 11*D* 超引力取非洛伦兹极限 [63]。预期该理论规范固定后会得到牛顿引力的 11*D* 超对称版本，这与本文中讨论的 3*D* 牛顿超引力理论思路一致。我们预期，按照相同方法也可以构造出 10*D* IIA 和 IIB 超引力的非洛伦兹版本。另一个需要关注的问题是本文所讨论的 10*D* 最小超引力理论的杂弦扩展。该理论应当包含杨-米尔斯和洛伦兹陈-西蒙斯项的非洛伦兹版本，这两个项在格林-施瓦茨反常消除机制中发挥了重要作用 [64]，预期这类反常消除在非洛伦兹情形下同样成立。

Finally, in the longer term, we hope that knowledge about the web of non-Lorentzian supergravity theories in diverse dimensions, as low-energy limits of non-Lorentzian string theories, will help to understand the role they might play in a holographic formulation for describing a new class of NR conformal field theories at the boundary along the lines of [47, 65, 66] .

最后，长期来看我们希望，作为非洛伦兹弦论的低能极限，不同维度下非洛伦兹超引力理论网络的相关知识能帮助我们理解这些理论在全息框架中可能发挥的作用，即沿着 [47, 65, 66] 的思路描述边界处一类新的 NR 共形场论。

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